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RECENT MATHEMATICAL PAPERS CONCERNING THE MOTIONS OF THE ATMOSPHERE.

PART I.

THE MOTIONS OF FLUIDS AND SOLIDS

ON THE EARTH'S SURFACE,

BY

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NOTE.

The publication of this compilation, with the accompanying notes, among the Professional Papers, is designed merely to bring it to the attention of scientific men, without endorsing any of the views or theories set forth.

INTRODUCTION.

Within the last twenty-five years, the motions of the atmosphere, as deduced from the principles of mechanics, have received the consideration of those who are desirous of establishing the meteorological science on analytical methods.

The object of the present paper is to place in the hand of the investigator and student the important writings on this subject, with such notes as may be of assistance to those who desire to arrive at the results and to understand the methods of research employed, without having to perform unnecessary labor in order to follow out each step of the analysis.

This paper will consist of two parts; reprints of mathematical essays of American and European authorship.

The first part will contain a reprint of the first and most important of the valuable mathematical essays on the motions of the atmosphere by Professor William Ferrel, United States Coast and Geodetic Survey.

Two more recent papers by Professor Ferrel published by the Coast and Geodetic Survey in 1875 and 1880, are so easily accessible that it is unnecessary to reprint them. These memoirs should be read and understood by the student of meteorology.

The second part will include the writings of several European mathematicians who have engaged in this important study. A short historical note will be appended to this part.

FRANK WALDO.

AUGUST, 1882.

NOTE TO PART I.

The following paper was first published by Professor William Ferrel, in "Runkle's Mathematical Monthly," during the years 1858 to 1860, and a small edition was reprinted in quarto pamphlet form in 1860.

The general course of reasoning employed by Professor Ferrel was first given by him in a popular article printed in the "Nashville Journal of Medicine and Surgery," October and November, 1856.

In the original paper, the $D_t x$ method of notation was employed, but as many of those who will read this paper are unaccustomed to this notation, used so extensively by Peirce, the more common form, $\frac{dx}{dt}$ has been used.

The original text of Professor Ferrel's paper is given in large type, while the matter in small type consists of such notes as will be of assistance to those wishing to consult the paper.

No notes have been appended to the chapter relating to the motion of solid bodies, as this is foreign to the meteorological topic under consideration.

Where there are several equations having the same number, individual numbers have been given to the parts by means of exponents. The parts of (7), for instance, are called (7^1), (7^2), (7^3).

THE

MOTIONS OF FLUIDS AND SOLIDS

ON THE EARTH'S SURFACE.

INTRODUCTION.

Some of the results contained in the following pages were published about two years ago, in an essay in the "Nashville Journal of Medicine and Surgery," edited by Professor W. K. Bowling, M. D., of Nashville, Tennessee. A small edition was also published in pamphlet form, and distributed by the Smithsonian Institution and myself, amongst various scientific men, libraries, and scientific associations, both in this country and in Europe. In that essay it was attempted to show that the depression of the atmosphere at the poles and the equator and the accumulation or bulging at the tropics, as indicated by barometric pressure, the gyratory motion of storms from right to left in the northern hemisphere, and the contrary way in the southern, and certain motions of oceanic currents, are necessary consequences of the modifying forces arising from the earth's rotation on its axis, and also that the observed flowing of the lower strata of the atmosphere in the middle latitudes towards the poles, contrary to the ordinary theory of the trade winds, is caused by the greater pressure of this accumulation of atmosphere at the tropics. It is believed that that essay was the first attempt to account for those remarkable phenomena by means of the modifying influence of the earth's rotation, and that it furnishes the only satisfactory explanation of them which has yet been given.

In that essay it was inconvenient to use any mathematical formulæ, and consequently the results merely, of only a partial and imperfect investigation of the subject were given; but it is thought that on account of the importance of the subject, it deserves a more thorough investigation. It is proposed, therefore, in the following pages, to go into a complete analytical investigation of the general motions of fluids surrounding the earth, and of projectiles at its surface, arising from disturbing forces and the earth's attraction, combined with the modifying forces arising from its rotation on its axis. We shall, accordingly, in the first section, investigate the general equations of motion relative to the earth's surface, applicable to both fluids and solids, and in the subsequent sections treat, first of the motions and figure of the whole or a part of a fluid surrounding the earth, upon the hypothesis that its motions are not resisted by the earth's surface, and then apply the results thus obtained to the explanation of the general motions of the atmosphere, the motions of storms or hurricanes, and the currents of the ocean. We shall also give a complete but concise treatise on projectiles, taking into account the effect of the earth's rotation.

We hope to be able, in this investigation, to give a satisfactory explanation of all the general motions of the atmosphere and of the ocean; the cause of the greater pressure of the atmosphere near the tropics than at the equator and the poles, and of the greater pressure generally in the northern hemisphere than in the southern; to account for the motion of all great storms in both hemispheres from the equator towards the poles in parabolic paths, and to completely establish their gyratory character; none of which phenomena have ever been satisfactorily accounted for by any of the usual theories, which do not take into account the influence of the earth's rotation.

SECTION I.

OF THE GENERAL EQUATIONS OF MOTION RELATIVE TO THE EARTH'S SURFACE.

1. Let x, y, z be three rectangular co-ordinates, having their origin at the centre of the earth, x corresponding with the axis of rotation. Also let

Ω be the potential of all the attractive forces of the earth,
P the pressure of the fluid, and
k its density.

Then $k\frac{d\,\Omega}{d\,x}$, $k\frac{d\,\Omega}{d\,y}$, and $k\frac{d\,\Omega}{d\,z}$ are the forces, in the reverse directions, for a unit of volume, arising from the earth's attraction, and $\frac{d\,P}{d\,x}$, $\frac{d\,P}{d\,y}$, and $\frac{d\,P}{d\,z}$, those arising from the pressure of the fluid, in the reverse directions, respectively, of x, y, and z; and we have for the equations of the absolute motions of the fluid, regarding the centre of the earth at rest,

$$(1)\qquad \begin{aligned} &\frac{d^2\,x}{d\,t^2}+\frac{d\,\Omega}{d\,x}+\frac{1}{k}\frac{d\,P}{d\,z}=0,\\ &\frac{d^2\,y}{d\,t^2}+\frac{d\,\Omega}{d\,y}+\frac{1}{k}\frac{d\,P}{d\,y}=0,\\ &\frac{d^2\,z}{d\,t^2}+\frac{d\,\Omega}{d\,z}+\frac{1}{k}\frac{d\,P}{d\,z}=0. \end{aligned}$$

Putting $P=0$, they are the equations of a projectile.

Ω is a function such that if the differentials be taken with respect to x, y, and z, we shall have the components of the force which arises from the attraction of all the particles of the earth, in the direction x, y, and z.

P is the pressure of the fluid within itself, it being a property that a solid cannot have. It is equal in all directions from any one part.

k is the density or mass of the fluid we are considering.

We must resolve all the forces, acting on the unit of volume of fluid, in the direction of the co-ordinates x, y, and z, and we get for the forces arising from the earth's attraction in the reverse directions, $k\frac{d\,\Omega}{d\,x}$, $k\frac{d\,\Omega}{d\,y}$, and $k\frac{d\,\Omega}{d\,z}$, the velocities generated in these directions in unit of time being $\frac{d\,\Omega}{d\,x}$, $\frac{d\,\Omega}{d\,y}$ and $\frac{d\,\Omega}{d\,z}$.

The forces arising from the pressure of the fluid in the reverse directions are $\frac{d\,P}{dx}$, $\frac{d\,P}{dy}$, and $\frac{d\,P}{dz}$.

From the dynamics of a particle we find that the acceleration of a particle referred to the axes x, y, and z are $\frac{d^2\,x}{d\,t^2}$, $\frac{d^2\,y}{d\,t^2}$, and $\frac{d^2\,z}{d\,t^2}$; and the forces corresponding to these accelerations are $k\frac{d^2\,x}{d\,t^2}$, $k\frac{d^2\,y}{d\,t^2}$, and $k\frac{d^2\,z}{d\,t^2}$.

We have, then, for the equation of the absolute motion of a unit of volume of the fluid, regarding the centre of the earth at rest,

$$\begin{aligned} &k\frac{d^2\,x}{d\,t^2}+k\frac{d\,\Omega}{d\,x}+\frac{d\,P}{d\,x}=0.\\ &k\frac{d^2\,y}{d\,t^2}+k\frac{d\,\Omega}{d\,y}+\frac{d\,P}{d\,y}=0.\\ &k\frac{d^2\,z}{d\,t^2}+k\frac{d\,\Omega}{d\,z}+\frac{d\,P}{d\,z}=0. \end{aligned}$$

dividing through by k, and we obtain the following equations:

$$(1)\qquad \begin{aligned} &(1^1)\quad \frac{d^2\,x}{d\,t^2}+\frac{d\,\Omega}{d\,x}+\frac{1}{k}\frac{d\,P}{d\,x}=0\\ &(1^2)\quad \frac{d^2\,y}{d\,t^2}+\frac{d\,\Omega}{d\,y}+\frac{1}{k}\frac{d\,P}{d\,y}=0\\ &(1^3)\quad \frac{d^2\,z}{d\,t^2}+\frac{d\,\Omega}{d\,z}+\frac{1}{k}\frac{d\,P}{d\,z}+0 \end{aligned}$$

Putting $P=0$, as in the case of a solid, and we have,

$$\frac{d^2\,x}{d\,t^2}+\frac{d\,\Omega}{d\,x}=0.\qquad \frac{d^2\,y}{d\,t^2}+\frac{d\,\Omega}{d\,y}=0.\qquad \frac{d^2\,z}{d\,t^2}+\frac{d\,\Omega}{d\,z}=0,$$

the equations of a projectile.

2. Let r be the distance from the earth's centre,
θ, the polar distance,
φ, the longitude, and
n, the angular velocity of the earth's rotation.

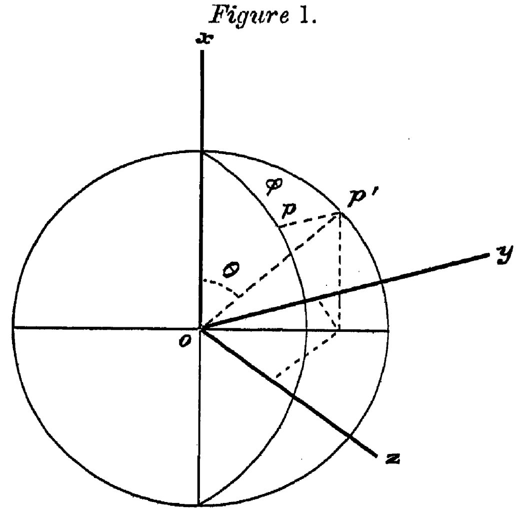
Then we have

$$(2)\qquad \begin{aligned} x &= r\cos\theta,\\ y &= r\sin\theta\cos(n\,t+\varphi) = r\sin\theta\cos\omega,\\ z &= r\sin\theta\sin(n\,t+\varphi) = r\sin\theta\sin\omega. \end{aligned}$$

by putting for brevity $n\,t+\varphi=\omega$.

The position of the ordinates y and z, and also the origin of the time t, being entirely arbitrary, they may be so taken as to make $\sin(n\,t+\varphi)$ vanish in the plane of x, y.

Figure 1.



Using these values of x, y, and z in equations (1), we obtain equations in which the first derivatives of r, θ, and φ represent the motions of the fluid or projectile relative to the earth's surface.

From Analytic Geometry of three dimensions we have,

(2^1) $x = r\cos\theta$.

(2^2) $y = r\sin\theta\cos(n\,t+\phi) = r\sin\theta\cos\omega$.

(2^3) $z = r\sin\theta\sin(n\,t+\phi) = r\sin\theta\sin\omega$.

Where t = the time given in seconds, and n and ϕ have the above-given signification.

The annexed diagram shows the relative position of the plane taken so that the $\sin(n\,t+\phi)$ will vanish in the plane $x\,y$.

3. Taking the first derivatives of (2) with regard to t, we get

$$\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \sin\theta\cos\omega\frac{dr}{dt} + r\cos\theta\cos\omega\frac{d\theta}{dt} - r\sin\theta\sin\omega\frac{d\omega}{dt},$$

$$\frac{dz}{dt} = \sin\theta\sin\omega\frac{dr}{dt} + r\cos\theta\sin\omega\frac{d\theta}{dt} + r\sin\theta\cos\omega\frac{d\omega}{dt}.$$

Taking the second derivatives, we get

$$\frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2},$$

$$(3)\qquad \begin{aligned} \frac{d^2y}{dt^2} = {}& \sin\theta\cos\omega\frac{d^2r}{dt^2} + 2\cos\theta\cos\omega\frac{dr}{dt}\frac{d\theta}{dt} - 2\sin\theta\sin\omega\frac{d\omega}{dt}\frac{dr}{dt} + r\cos\theta\cos\omega\frac{d^2\theta}{dt^2}\\ & - r\sin\theta\cos\omega\left(\frac{d\theta}{dt}\right)^2 - 2r\cos\theta\sin\omega\frac{d\omega}{dt}\frac{d\theta}{dt} - r\sin\theta\sin\omega\frac{d^2\varphi}{dt^2} - r\sin\theta\cos\omega\left(\frac{d\omega}{dt}\right)^2.\\ \frac{d^2z}{dt^2} = {}& \sin\theta\sin\omega\frac{d^2r}{dt^2} + 2\cos\theta\sin\omega\frac{dr}{dt}\frac{d\theta}{dt} + 2\sin\theta\cos\omega\frac{d\omega}{dt}\frac{dr}{dt} + r\cos\theta\sin\omega\frac{d^2\theta}{dt^2}\\ & - r\sin\theta\sin\omega\left(\frac{d\theta}{dt}\right)^2 + 2r\cos\theta\cos\omega\frac{d\omega}{dt}\frac{d\theta}{dt} + r\sin\theta\cos\omega\frac{d^2\varphi}{dt^2} - r\sin\theta\sin\omega\left(\frac{d\omega}{dt}\right)^2. \end{aligned}$$

Take the first derivative of (2) with regard to t;

(2^1). $x = r\cos\theta$

$$\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}.$$

(2^2). $y = r\sin\theta\cos\omega$

$$\frac{dy}{dt} = \sin\theta\cos\omega\frac{dr}{dt} + r\cos\theta\cos\omega\frac{d\theta}{dt} - r\sin\theta\sin\omega\frac{d\omega}{dt}.$$

(2^3). $z = r\sin\theta\sin\omega$

$$\frac{dz}{dt} = \sin\theta\sin\omega\frac{dr}{dt} + r\cos\theta\sin\omega\frac{d\theta}{dt} + r\sin\theta\cos\omega\frac{d\omega}{dt}.$$

Take the second derivatives of (2) with regard to t.

The first derivative of (2^1) is,

$$\frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}.\quad\text{Then}$$

$$(3^1)\qquad \frac{d^2x}{dt^2} = \cos\theta\frac{d^2r}{dt^2} - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}.$$

The first derivative of (2^2) is

$$\frac{dy}{dt} = \sin\theta\cos\omega\frac{dr}{dt} + r\cos\theta\cos\omega\frac{d\theta}{dt} - r\sin\theta\sin\omega\frac{d\omega}{dt}.\quad\text{Then,}$$

$$\frac{d^2 y}{d t^2} = \sin\theta\cos\omega\frac{d^2 r}{d t^2} - \sin\theta\sin\omega\frac{d\omega}{d t}\frac{d r}{d t} + \cos\theta\cos\omega\frac{d\theta}{d t}\frac{d r}{d t} + r\cos\theta\cos\omega\frac{d^2\theta}{d t^2} - r\cos\theta\sin\omega\frac{d\omega}{d t}\frac{d\theta}{d t}$$
$$- r\sin\theta\cos\omega\left(\frac{d\theta}{d t}\right)^2 + \cos\theta\cos\omega\frac{d r}{d t}\frac{d\theta}{d t} - r\sin\theta\sin\omega\frac{d^2\omega}{d t^2} - r\sin\theta\cos\omega\left(\frac{d\omega}{d t}\right)^2$$
$$- r\cos\theta\sin\omega\frac{d\theta}{d t}\frac{d\omega}{d t} - \sin\theta\sin\omega\frac{d\omega}{d t}\frac{d r}{d t}.$$

From article 2, we have,
$\omega = n t + \phi$.
Then $d\omega = n\, d t + d\phi$

$$\frac{d\omega}{d t} = n + \frac{d\phi}{d t}. \quad \frac{d^2\omega}{d t^2} = \frac{d^2\phi}{d t^2}. \quad n \text{ being a constant.}$$

$$(3^2) \quad \frac{d^2 y}{d t^2} = \sin\theta\cos\omega\frac{d^2 r}{d t^2} + 2\cos\theta\cos\omega\frac{d r}{d t}\frac{d\theta}{d t} - 2\sin\theta\sin\omega\frac{d\omega}{d t}\frac{d r}{d t} + r\cos\theta\cos\omega\frac{d^2\theta}{d t^2} - r\sin\theta\cos\omega\left(\frac{d\theta}{d t}\right)^2$$
$$- 2r\cos\theta\sin\omega\frac{d\omega}{d t}\frac{d\theta}{d t} - r\sin\theta\sin\omega\frac{d^2\phi}{d t^2} - r\sin\theta\cos\omega\left(\frac{d\omega}{d t}\right)^2.$$

It will be noticed that $\frac{d^2\phi}{d t^2}$ is substituted for $\frac{d^2\omega}{d t^2}$ in the next to the last term of the second member of the equation.

The first derivative of (2^3) is,

$$\frac{d z}{d t} = \sin\theta\sin\omega\frac{d r}{d t} + r\cos\theta\sin\omega\frac{d\theta}{d t} + r\sin\theta\cos\omega\frac{d\omega}{d t}.$$

Then
$$\frac{d^2 z}{d t^2} = \sin\theta\sin\omega\frac{d^2 r}{d t^2} + \sin\theta\cos\omega\frac{d\omega}{d t}\frac{d r}{d t} + \cos\theta\sin\omega\frac{d\theta}{d t}\frac{d r}{d t} + r\cos\theta\sin\omega\frac{d^2\theta}{d t^2} + r\cos\theta\cos\omega\frac{d\omega}{d t}\frac{d\theta}{d t}$$
$$- r\sin\theta\sin\omega\left(\frac{d\theta}{d t}\right)^2 + \cos\theta\sin\omega\frac{d r}{d t}\frac{d\theta}{d t} + r\sin\theta\cos\omega\frac{d^2\omega}{d t^2} - r\sin\theta\sin\omega\left(\frac{d\omega}{d t}\right)^2$$
$$+ r\cos\theta\cos\omega\frac{d\theta}{d t}\frac{d\omega}{d t} + \sin\theta\cos\omega\frac{d r}{d t}\frac{d\omega}{d t}.$$

$$(3^3) \quad \frac{d^2 z}{d t^2} = \sin\theta\sin\omega\frac{d^2 r}{d t^2} + 2\cos\theta\sin\omega\frac{d r}{d t}\frac{d\theta}{d t} + 2\sin\theta\cos\omega\frac{d\omega}{d t}\frac{d r}{d t} + r\cos\theta\sin\omega\frac{d^2\theta}{d t^2} - r\sin\theta\sin\omega\left(\frac{d\theta}{d t}\right)^2$$
$$+ 2r\cos\theta\cos\omega\frac{d\omega}{d t}\frac{d\theta}{d t} + r\sin\theta\cos\omega\frac{d^2\phi}{d t^2} - r\sin\theta\sin\omega\left(\frac{d\omega}{d t}\right)^2.$$

Here again $\frac{d^2\phi}{d t^2}$ is substituted for $\frac{d^2\omega}{d t^2}$ in the next to the last term.

Since x, y, and z are functions of r, θ, and φ, we must put

$$(4) \quad \begin{aligned} \frac{d\Omega}{d x} &= \frac{d\Omega}{d r}\frac{d r}{d x} + \frac{d\Omega}{d\theta}\frac{d\theta}{d x} + \frac{d\Omega}{d\varphi}\frac{d\varphi}{d x}, \\ \frac{d\Omega}{d y} &= \frac{d\Omega}{d r}\frac{d r}{d y} + \frac{d\Omega}{d\theta}\frac{d\theta}{d y} + \frac{d\Omega}{d\varphi}\frac{d\varphi}{d y}, \\ \frac{d\Omega}{d z} &= \frac{d\Omega}{d r}\frac{d r}{d z} + \frac{d\Omega}{d\theta}\frac{d\theta}{d z} + \frac{d\Omega}{d\varphi}\frac{d\varphi}{d z}. \end{aligned}$$

Now we have

$$r^2 = x^2 + y^2 + z^2,$$
$$\tan\theta = \frac{\sqrt{y^2 + z^2}}{x},$$
$$\tan\omega = \frac{z}{y}.$$

Hence,

$$\frac{d r}{d x} = \frac{x}{r} = \cos\theta,$$
$$\frac{d r}{d y} = \frac{y}{r} = \sin\theta\cos\omega,$$
$$\frac{d r}{d z} = \frac{z}{r} = \sin\theta\sin\omega,$$
$$\frac{d\theta}{d x} = -\frac{\sqrt{y^2 + z^2}}{r^2} = -\frac{\sin\theta}{r},$$
$$\frac{d\theta}{d y} = \frac{x y}{r^2\sqrt{y^2 + z^2}} = \frac{\cos\theta\cos\omega}{r},$$
$$\frac{d\theta}{d z} = \frac{x z}{r^2\sqrt{y^2 + z^2}} = \frac{\cos\theta\sin\omega}{r},$$

$$\frac{d\varphi}{dx}=0,$$
$$\frac{d\varphi}{dy}=\frac{-z}{y^2+z^2}=\frac{-\sin\omega}{r\sin\theta},$$
$$\frac{d\varphi}{dz}=\frac{y}{y^2+z^2}=\frac{\cos\omega}{r\sin\theta}.$$

We have $r^2=x^2+y^2+z^2$, from analytic geometry.

Then $r^2-x^2=y^2+z^2$. $\frac{r^2}{x^2}-1=\frac{y^2+z^2}{x^2}$. Substituting in this equation the value $\frac{x}{r}=\cos\theta$, which is found from $x=r\cos\theta$ (2^1), we have,

$$\frac{1}{\cos^2\theta}-1=\frac{y^2+z^2}{x^2}. \qquad \frac{1-\cos^2\theta}{\cos^2\theta}=\frac{y^2+z^2}{x^2}. \qquad \frac{\sin^2\theta}{\cos^2\theta}=\frac{y^2+z^2}{x^2}. \qquad \operatorname{Tan}\theta=\frac{\sqrt{y^2+z^2}}{x}.$$

Also dividing (2^3) by (2^2) we have, $\tan\omega=\frac{z}{y}$.

From $r^2=x^2+y^2+z^2$ we must get the partial differential co-efficients:

$$2\,r\,dr=2\,x\,dx. \qquad \frac{dr}{dx}=\frac{x}{r}=\cos\theta.$$
$$2\,r\,dr=2\,y\,dy. \qquad \frac{dr}{dy}=\frac{y}{r}=\sin\theta\cos\omega.$$
$$2\,r\,dr=2\,z\,dz. \qquad \frac{dr}{dz}=\frac{z}{r}=\sin\theta\sin\omega.$$

The values of $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$ are obtained directly from (2).

We have $\cos\theta=\frac{x}{r}$, then $\cos^2\theta=\frac{x^2}{r^2}$.

From differential calculus $d\tan\theta=\frac{d\theta}{\cos^2\theta}$, but $\tan\theta=\frac{\sqrt{y^2+z^2}}{x}$ (as previously determined).

Substituting the value of $\tan\theta$, also of $\cos^2\theta$ in the above differential equation, we have

$$d\left(\frac{\sqrt{y^2+z^2}}{x}\right)=\frac{d\theta\, r^2}{x^2}, \text{ or, } -\frac{(\sqrt{y^2+z^2})\,dx}{x^2}=\frac{d\theta\, r^2}{x^2}.$$
$$-\sqrt{y^2+z^2}\,dx=d\theta\, r^2. \quad \frac{d\theta}{dx}=-\frac{\sqrt{y^2+z^2}}{r^2}.$$

By trigonometry we get, $\sin\theta=\tan\theta\cos\theta$, $\therefore \sin\theta=\frac{\sqrt{y^2+z^2}}{x}\,\frac{x}{r}=\frac{\sqrt{y^2+z^2}}{r}$. Substituting this in the preceding equation, we get, $\frac{d\theta}{dx}=-\frac{\sin\theta}{r}$. $\quad d\tan\theta=\frac{d\theta}{\cos^2\theta}$. $\quad d\left(\frac{\sqrt{y^2+z^2}}{x}\right)=\frac{d\theta}{\cos^2\theta}=\frac{d\theta}{\frac{x^2}{r^2}} \quad \frac{\frac{2\,y\,dy}{2\sqrt{y^2+z^2}}}{x}=\frac{d\theta\, r^2}{x^2}.$

$$\frac{y\,dy}{\sqrt{y^2+z^2}}=\frac{d\theta\, r^2}{x}. \quad \frac{d\theta}{dy}=\frac{x\,y}{r^2\sqrt{y^2+z^2}}.$$

We also have, $\frac{d\theta}{dy}=\frac{x}{r\sqrt{y^2+z^2}}\cos\theta. \qquad \cos\omega\,\frac{y}{r\sin\theta}=\frac{y}{r\frac{\sqrt{y^2+z^2}}{r}}=\frac{y}{\sqrt{y^2+z^2}}.$

$$\frac{d\theta}{dy}=\frac{1}{r}\,\frac{x}{r}\,\frac{y}{\sqrt{y^2+z^2}}=\frac{1}{r}\cos\theta\cos\omega.$$

$$d\tan\theta=\frac{d\theta}{\cos^2\theta}=d\left(\frac{\sqrt{y^2+z^2}}{x}\right). \qquad \frac{d\theta}{\cos^2\theta}=\frac{x\,\frac{1}{2}(y^2+z^2)^{-\frac{1}{2}}2\,z\,dz}{x^2}. \qquad \frac{d\theta}{\cos^2\theta}=\frac{\frac{x\,z}{(y^2+z^2)^{\frac{1}{2}}}\,dz}{x^2}.$$

$$\frac{d\theta}{dz}=\frac{\frac{\cos^2\theta\, x\, z}{(y^2+z^2)^{\frac{1}{2}}}}{x^2}=\left(\frac{x^2}{r^2}\right)\frac{\frac{x\,z}{(y^2+z^2)^{\frac{1}{2}}}}{x^2}=\frac{x\,z}{r^2(y^2+z^2)^{\frac{1}{2}}}=\frac{x\,z}{r^2\sqrt{y^2+z^2}}.$$

Also, $\frac{d\theta}{dz}=\frac{x\,z}{r^2\sqrt{y^2+z^2}}=\frac{1}{r}\,\frac{z}{r}\,\frac{x}{\sqrt{y^2+z^2}}=\frac{1}{r}\sin\theta\sin\omega\,\frac{1}{\tan\theta}.$

Then, $\frac{d\theta}{dz}=\frac{1}{r}\cos\theta\sin\omega.$

From (2),

$\tan\omega=\frac{z}{y}$. Then $\omega=\tan^{-1}\frac{z}{y}=\phi+n\,t$. $\quad \phi=\tan^{-1}\frac{z}{y}-n\,t$. Differentiating with respect to x we have

$$\frac{d\phi}{dx}=0.$$

$$\operatorname{Tan}\omega=\frac{z}{y}.\qquad \frac{d\,\omega}{\cos^2\omega}=-\frac{z\,d\,y}{y^2}.\qquad \frac{d\,\omega}{d\,y}=\frac{-z\cos^2\omega}{y^2}=\frac{-z}{y^2}\left(\frac{y^2}{r^2\sin^2\theta}\right)=z\,\frac{1}{r^2\left(1-\frac{x^2}{r^2}\right)}=-\frac{z}{r^2-x^2}.$$

But $r^2-x^2=y^2+z^2\ \therefore\ \dfrac{d\,\omega}{d\,y}=-\dfrac{z}{y^2+z^2}$.

Also, $\dfrac{-z}{y^2+z^2}=\dfrac{-r\sin\theta\sin\omega}{(r^2\sin^2\theta)}=\dfrac{-\sin\omega}{r\sin\theta}$.

$\operatorname{Tan}\omega=\dfrac{z}{y}$. Then, $\dfrac{d\,\omega}{\cos^2\omega}=\dfrac{y\,d\,z}{y^2}$. $\dfrac{d\,\omega}{d\,z}=\dfrac{1}{y}(\cos^2\omega)=\dfrac{1}{y}\left(\dfrac{y^2}{r^2\sin^2\theta}\right)$. $\dfrac{d\,\omega}{d\,z}=\dfrac{y}{y^2+z^2}$.

Also. $\dfrac{y}{y^2+z^2}=\dfrac{r\sin\theta\cos\omega}{r^2\sin^2\theta\,(\cos^2\omega+\sin^2\omega)}=\dfrac{\cos\omega}{r\sin\theta}$.

By means of these equations, equations (4) become

$$\frac{d\,\Omega}{d\,x}=\frac{d\,\Omega}{d\,r}\cos\theta-\frac{d\,\Omega}{d\,\theta}\,\frac{\sin\theta}{r},$$

$$(5)\qquad \frac{d\,\Omega}{d\,y}=\frac{d\,\Omega}{d\,r}\sin\theta\cos\omega+\frac{d\,\Omega}{d\,\theta}\,\frac{\cos\theta\cos\omega}{r}-\frac{d\,\Omega}{d\,\varphi}\,\frac{\sin\omega}{r\sin\theta}.$$

$$\frac{d\,\Omega}{d\,z}=\frac{d\,\Omega}{d\,r}\sin\theta\sin\omega+\frac{d\,\Omega}{d\,\theta}\,\frac{\cos\theta\sin\omega}{r}+\frac{d\,\Omega}{d\,\varphi}\,\frac{\cos\varphi}{r\sin\theta}.$$

In the same manner we obtain

$$\frac{d\,P}{d\,x}=\frac{d\,P}{d\,r}\cos\theta-\frac{d\,P}{d\,\theta}\,\frac{\sin\theta}{r},$$

$$(6)\qquad \frac{d\,P}{d\,y}=\frac{d\,P}{d\,r}\sin\theta\cos\omega+\frac{d\,P}{d\,\theta}\,\frac{\cos\theta\cos\omega}{r}-\frac{d\,P}{d\,\varphi}\,\frac{\sin\omega}{r\sin\theta}.$$

$$\frac{d\,P}{d\,z}=\frac{d\,P}{d\,r}\sin\theta\sin\omega+\frac{d\,P}{d\,\theta}\,\frac{\cos\theta\sin\omega}{r}+\frac{d\,P}{d\,\varphi}\,\frac{\cos\omega}{r\sin\theta}.$$

Substituting the values of the first members of equations (3), (5), and (6) in equations (1), and multiplying them respectively by $\cos\theta$, $\sin\theta\cos\omega$, $\sin\theta\sin\omega$, and adding, we obtain the first of the following equations. Again, multiplying them respectively by $r\sin\theta$, $-r\cos\theta\cos\omega$, and $-r\cos\theta\sin\omega$, and adding, we obtain the second of these equations. Finally, multiplying the last two respectively by $r\sin\theta\sin\omega$, and $-r\sin\theta\cos\omega$, and adding, we get the last of the following equations:

It will be noticed that the three terms in equations (1) are the first members respectively in the equations (3), (5), and (6) $\times\ \frac{1}{k}$, so that the substitution referred to above is easily made. It consists simply in adding algebraically the second members of the corresponding equations in (3), (5), and (6) $\times\ \frac{1}{k}$ respectively, and placing the sums equal to 0. This will give us three equations, (a), (b), and (c): The first of which is to be multiplied by $\cos\theta$; the second of which is to be multiplied by $\sin\theta\cos\omega$; the third of which is to be multiplied by $\sin\theta\sin\omega$. The sum is then to be taken and we have (7^1).

The same three equations, (a), (b), and (c), are then to be operated upon in the following manner: The first is to be multiplied by $r\sin\theta$; the second by $-r\cos\theta\cos\omega$: and the third by $-r\cos\theta\sin\omega$. The sum is then to be taken and we have (7^2).

The equations (b) and (c) are to be treated as follows: The first is to be multiplied by $r\sin\theta\sin\omega$, and the second by $-r\sin\theta\cos\omega$. The sum is then to be taken and we have (7^3).

These multiplications and additions are rather tedious, but the shortest way seems to be as follows:

Multiply the terms containing the same differential coefficients in the equations (a), (b), and (c) by their respective multipliers for determining (7^1) and add the products, affixing to the sum the common differential coefficient. The sum of all the products so found, reduced and simplified, will of course give (7^1).

Similarly, using the respective multiplters, we obtain (7^2).

In like manner, using the respective multipliers, we obtain (7^3).

$$(a).\qquad \cos\theta\frac{d^2r}{dt^2}-2\sin\theta\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}-r\cos\theta\left(\frac{d\,\theta}{d\,t}\right)^2-r\sin\theta\frac{d^2\theta}{dt^2}+\cos\theta\frac{d\,\Omega}{d\,r}-\frac{\sin\theta}{r}\frac{d\,\Omega}{d\,\theta}+\frac{1}{k}\cos\theta\frac{d\,P}{d\,r}-\frac{1}{k}\frac{\sin\theta}{r}\frac{d\,P}{d\,\theta}=0.$$

$$(b).\qquad \left\{\begin{array}{l}\sin\theta\cos\omega\dfrac{d^2r}{dt^2}+2\cos\theta\cos\omega\dfrac{d\,r}{d\,t}\dfrac{d\,\theta}{d\,t}-2\sin\theta\sin\omega\dfrac{d\,\omega}{d\,t}\dfrac{d\,r}{d\,t}+r\cos\theta\cos\omega\dfrac{d^2\theta}{dt^2}\\ -r\sin\theta\cos\omega\left(\dfrac{d\,\theta}{d\,t}\right)^2-2r\cos\theta\sin\omega\dfrac{d\,\omega}{d\,t}\dfrac{d\,\theta}{d\,t}-r\sin\theta\sin\omega\dfrac{d^2\phi}{dt^2}-r\sin\theta\cos\omega\left(\dfrac{d\,\omega}{d\,t}\right)^2\end{array}\right\}+\sin\theta\cos\omega\frac{d\,\Omega}{d\,r}$$

$$+\frac{\cos\theta\cos\omega}{r}\frac{d\,\Omega}{d\,\theta}-\frac{\sin\omega}{r\sin\theta}\frac{d\,\Omega}{d\,\phi}+\frac{1}{k}\sin\theta\cos\omega\frac{d\,P}{d\,r}+\frac{1}{k}\frac{\cos\theta\cos\omega}{r}\frac{d\,P}{d\,\theta}-\frac{1}{k}\frac{\sin\omega}{r\sin\theta}\frac{d\,P}{d\,\phi}=0.$$

$$(c).\quad \left\{\begin{array}{l}\sin\theta\sin\omega\dfrac{d^2 r}{d t^2}+2\cos\theta\sin\omega\dfrac{d r}{d t}\dfrac{d\theta}{d t}+2\sin\theta\cos\omega\dfrac{d\omega}{d t}\dfrac{d r}{d t}+r\cos\theta\sin\omega\dfrac{d^2\theta}{d t^2}\\ -r\sin\theta\sin\omega\left(\dfrac{d\theta}{d t}\right)^2+2r\cos\theta\cos\omega\dfrac{d\omega}{d t}\dfrac{d\theta}{d t}+r\sin\theta\cos\omega\dfrac{d^2\phi}{d t^2}-r\sin\theta\sin\omega\left(\dfrac{d\omega}{d t}\right)^2\end{array}\right\}+\sin\theta\sin\omega\frac{d\Omega}{d r}$$

$$+\frac{\cos\theta\sin\omega}{r}\frac{d\Omega}{d\theta}+\frac{\cos\omega}{r\sin\theta}\frac{d\Omega}{d\phi}+\frac{1}{k}\sin\theta\sin\omega\frac{d P}{d r}+\frac{1}{k}\frac{\cos\theta\sin\omega}{r}\frac{d P}{d\theta}+\frac{1}{k}\frac{\cos\omega}{r\sin\theta}\frac{d P}{d\phi}=0.$$

Work for determining (7^1)

$$(\cos^2\theta+\sin^2\theta\cos^2\omega+\sin^2\theta\sin^2\omega)\frac{d^2 r}{d t^2}=(\cos^2\theta+\sin^2\theta(\cos^2\omega+\sin^2\omega))\frac{d^2 r}{d t^2}=(\cos^2\theta+\sin^2\theta(1))\frac{d^2 r}{d t^2}=\frac{d^2 r}{d t^2}.$$

$$(-2\sin\theta\cos\theta+2\sin\theta\cos\theta\cos^2\omega+2\sin\theta\cos\theta\sin^2\omega)\frac{d r}{d t}\frac{d\theta}{d t}$$

$$=(-2\sin\theta\cos\theta+2\sin\theta\cos\theta(\cos^2\omega+\sin^2\omega))\frac{d r}{d t}\frac{d\theta}{d t}=0\frac{d r}{d t}\frac{d\theta}{d t}=0.$$

$$(-r\sin\theta\cos\theta+r\sin\theta\cos\theta\cos^2\omega+r\sin\theta\cos\theta\sin^2\omega)\frac{d^2\theta}{d t^2}$$

$$=(-r\sin\theta\cos\theta+r\sin\theta\cos\theta)\frac{d^2\theta}{d t^2}=0\frac{d^2\theta}{d t^2}=0.$$

$$(-r\sin^2\theta\sin\omega\cos\omega+r\sin^2\theta\sin\omega\cos\omega)\frac{d^2\phi}{d t^2}=0\frac{d^2\phi}{d t^2}=0.$$

$$(-r\cos^2\theta-r\sin^2\theta\cos^2\omega-r\sin^2\theta\sin^2\omega)\left(\frac{d\theta}{d t}\right)^2=(-r\cos^2\theta-r\sin^2\theta)\left(\frac{d\theta}{d t}\right)^2=-r(1)\left(\frac{d\theta}{d t}\right)^2.$$

$$(-r\sin^2\theta\cos^2\omega-r\sin^2\theta\sin^2\omega)\left(\frac{d\omega}{d t}\right)^2=(-r\sin^2\theta)(1)\left(\frac{d\omega}{d t}\right)^2.$$

$$(-2\sin^2\theta\sin\omega\cos\omega+2\sin^2\theta\sin\omega\cos\omega)\frac{d\omega}{d t}\frac{d r}{d t}=0\frac{d\omega}{d t}\frac{d r}{d t}=0.$$

$$(-2r\sin\theta\cos\theta\sin\omega\cos\omega+2r\sin\theta\cos\theta\sin\omega\cos\omega)\frac{d\omega}{d t}\frac{d\theta}{d t}=0\frac{d\omega}{d t}\frac{d\theta}{d t}=0.$$

$(\cos^2\theta+\sin^2\theta\cos^2\omega+\sin^2\theta\sin^2\omega)\dfrac{d\Omega}{d r}=(\cos^2\theta+\sin^2\theta)\dfrac{d\Omega}{d r}=\dfrac{d\Omega}{d r}.$ We have also the corresponding term $\dfrac{1}{k}\dfrac{d P}{d r}$.

$$\left(\frac{-\sin\theta\cos\theta}{r}+\frac{\sin\theta\cos\theta\cos^2\omega}{r}+\frac{\sin\theta\cos\theta\sin^2\omega}{r}\right)\frac{d\Omega}{d\theta}=\left(\frac{-\sin\theta\cos\theta}{r}+\frac{\sin\theta\cos\theta}{r}\right)\frac{d\Omega}{d\theta}=0\frac{d\Omega}{d\theta}=0.$$

We have also the corresponding term $0\dfrac{1}{k}\dfrac{d P}{d\theta}=0.$

$\left(\dfrac{-\sin\omega\cos\omega}{r}+\dfrac{\sin\omega\cos\omega}{r}\right)\dfrac{d\Omega}{d\phi}=0\dfrac{d\Omega}{d\phi}=0.$ We have also the corresponding term $0\dfrac{1}{k}\dfrac{d P}{d\phi}=0.$

Taking the sum of these results and placing it $=0$, we have,

$$\frac{d^2 r}{d t^2}-r\left(\frac{d\theta}{d t}\right)^2-r\sin^2\theta\left(\frac{d\omega}{d t}\right)^2+\frac{d\Omega}{d r}+\frac{1}{k}\frac{d P}{d r}=0.\quad \frac{1}{k}\frac{d P}{d r}=-\frac{d^2 r}{d t^2}+r\left(\frac{d\theta}{d t}\right)^2+r\sin^2\theta\left(\frac{d\omega}{d t}\right)^2-\frac{d\Omega}{d r}.$$

But $\omega=\phi+n t$, and $d\omega=n\,d t+d\phi$. $\quad\dfrac{d\omega}{d t}=n+\dfrac{d\phi}{d t}$.

Also $r\sin^2\theta\left(\dfrac{d\omega}{d t}\right)^2=r\sin^2\theta\dfrac{d\omega}{d t}\dfrac{d\omega}{d t}$. Substituting the value of $\dfrac{d\omega}{d t}$ in this we get,

$r\sin^2\theta\left(n^2+2n\dfrac{d\phi}{d t}+\left(\dfrac{d\phi}{d t}\right)^2\right)=r\sin^2\theta\, n^2+r\sin^2\theta\left(2n+\dfrac{d\phi}{d t}\right)\dfrac{d\phi}{d t}$, which, as $2n+\dfrac{d\phi}{d t}=\dfrac{d\omega}{d t}+n$, is equal to $r\sin^2\theta\, n^2+r\sin^2\theta\left(n+\dfrac{d\omega}{d t}\right)\dfrac{d\phi}{d t}$.

$$(7^1)\ \therefore\ \frac{1}{k}\frac{d P}{d r}=-\frac{d^2 r}{d t^2}+r\left(\frac{d\theta}{d t}\right)^2+r\sin^2\theta\left(n+\frac{d\omega}{d t}\right)\frac{d\phi}{d t}+r n^2\sin^2\theta-\frac{d\Omega}{d r}.$$

Work for determining (7^2).

$$(r\sin\theta\cos\theta-r\sin\theta\cos\theta\cos^2\omega-r\sin\theta\cos\theta\sin^2\omega)\frac{d^2 r}{d t^2}=(r\sin\theta\cos\theta-r\sin\theta\cos\theta(1))\frac{d^2 r}{d t^2}=0\frac{d^2 r}{d t^2}=0.$$

$$(2r\sin^2\theta-2r\cos^2\theta\cos^2\omega-2r\cos^2\theta\sin^2\omega)\frac{d r}{d t}\frac{d\theta}{d t}=(-2r\sin^2\theta-2r\cos^2\theta(1))\frac{d r}{d t}\frac{d\theta}{d t}=-2r(1)\frac{d r}{d t}\frac{d\theta}{d t}.$$

$$(-r^2\sin^2\theta-r^2\cos^2\theta\cos^2\omega-r^2\cos^2\theta\sin^2\omega)\frac{d^2\theta}{d t^2}=(-r^2\sin^2\theta-r^2\cos^2\theta(1))\frac{d^2\theta}{d t^2}=-r^2(1)\frac{d^2\theta}{d t^2}.$$

$(+r^2\sin\theta\cos\theta\sin\omega\cos\omega - r^2\sin\theta\cos\theta\sin\omega\cos\omega)\frac{d^2\phi}{d\,t^2} = 0\frac{d^2\phi}{d\,t^2} = 0.$

$(-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta\cos^2\omega + r^2\sin\theta\cos\theta\sin^2\omega)\left(\frac{d\,\theta}{d\,t}\right)^2$

$= (-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta\,(1))\left(\frac{d\,\theta}{d\,t}\right)^2 = 0\left(\frac{d\,\theta}{d\,t}\right)^2 = 0.$

$(+r^2\sin\theta\cos\theta\cos^2\omega + r^2\sin\theta\cos\theta\sin^2\omega)\left(\frac{d\,\omega}{d\,t}\right)^2 = r^2\sin\theta\cos\theta\,(1)\left(\frac{d\,\omega}{d\,t}\right)^2.$

$(+2\,r\sin\theta\cos\theta\sin\omega\cos\omega - 2\,r\sin\theta\cos\theta\sin\omega\cos\omega)\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} = 0\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} = 0.$

$(+2\,r^2\cos^2\theta\sin\omega\cos\omega - 2\,r^2\cos^2\theta\sin\omega\cos\omega)\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t} = 0\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t} = 0.$

$(r\sin\theta\cos\theta - r\sin\theta\cos\theta\cos^2\omega - r\sin\theta\cos\theta\sin^2\omega)\frac{d\,\Omega}{d\,r} = (r\sin\theta\cos\theta - r\sin\theta\cos\theta\,(1)\,)\frac{d\,\Omega}{d\,r} = 0\frac{d\,\Omega}{d\,r} = 0.$

We have also the corresponding term $0\,\frac{1}{k}\frac{d\,P}{d\,r} = 0.$

$\left(\frac{-r\sin^2\theta}{r}\frac{-r\cos^2\theta\cos^2\omega}{r}\frac{-r\cos^2\theta\sin^2\omega}{r}\right)\frac{d\,\Omega}{d\theta} = (-\sin^2\theta - \cos^2\theta\,(1)\,)\frac{d\,\Omega}{d\theta} = -(1)\frac{d\,\Omega}{d\theta}.$ We have also the the corresponding term $-\frac{1}{k}\frac{d\,P}{d\,\theta}.$

$\left(+\frac{r\cos\theta\sin\omega\cos\omega}{r\sin\theta} - \frac{r\cos\theta\sin\omega\cos\omega}{r\sin\theta}\right)\frac{d\,\Omega}{d\phi} = 0\frac{d\,\Omega}{d\phi} = 0.$ We have also the corresponding term $0\,\frac{1}{k}\frac{d\,P}{d\,\phi} = 0.$

Taking the sum of these results and placing it $= 0$ we have,

$$\frac{1}{k}\frac{d\,P}{d\,\theta} = -r^2\frac{d^2\,\theta}{d\,t^2} - 2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} + r^2\sin\theta\cos\theta\left(\frac{d\,\omega}{d\,t}\right)^2 - \frac{d\,\Omega}{d\,\theta}.$$

But, $r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\frac{d\,\omega}{d\,t} = r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\left(\frac{d\,\phi}{d\,t} + n\right) = r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\frac{d\,\phi}{d\,t} + r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\,n$

$= r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\frac{d\,\phi}{d\,t} + r^2\sin\theta\cos\theta\left(\frac{d\,\phi}{d\,t} + n\right)n = r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\frac{d\,\phi}{d\,t} + r^2\sin\theta\cos\theta\,n\,\frac{d\,\phi}{d\,t}$

$+ r^2\sin\theta\cos\theta\,n^2.$

Substituting this value in the equation given above we have,

$$(7^2)\quad \frac{1}{k}\frac{d\,P}{d\,\theta} - r^2\frac{d^2\,\theta}{d\,t^2} - 2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} + r^2\sin\theta\cos\theta\left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t} + r^2 n^2\sin\theta\cos\theta - \frac{d\,\Omega}{d\,\theta}.$$

Work for determining (7^3).

$(r\sin^2\theta\sin\omega\cos\omega - r\sin^2\theta\sin\omega\cos\omega)\frac{d^2\,r}{d\,t^2} = 0\frac{d^2\,r}{d\,t^2} = 0.$

$(2\,r\sin\theta\cos\theta\sin\omega\cos\omega - 2\,r\sin\theta\cos\theta\sin\omega\cos\omega)\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} = 0\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} = 0.$

$(r^2\sin\theta\cos\theta\sin\omega\cos\omega - r^2\sin\theta\cos\theta\sin\omega\cos\omega)\frac{d^2\,\theta}{d\,t^2} = 0\frac{d^2\,\theta}{d\,t^2} = 0.$

$(-r^2\sin^2\theta\sin^2\omega - r^2\sin^2\theta\cos^2\omega)\frac{d^2\,\phi}{d\,t^2} = -r^2\sin^2\theta\,(1)\frac{d^2\,\phi}{d\,t^2}.$

$(-r^2\sin^2\theta\sin\omega\cos\omega + r^2\sin^2\theta\sin\omega\cos\omega)\left(\frac{d\,\theta}{d\,t}\right)^2 = 0\left(\frac{d\,\theta}{d\,t}\right)^2 = 0.$

$(-r^2\sin^2\theta\sin\omega\cos\omega + r^2\sin^2\theta\sin\omega\cos\omega)\left(\frac{d\,\omega}{d\,t}\right)^2 = 0\left(\frac{d\,\omega}{d\,t}\right)^2.$

$(-2\,r\sin^2\theta\sin^2\omega - 2\,r\sin^2\theta\cos^2\omega)\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} = -2\,r\sin^2\theta\,(1)\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t}.$

$(-r\sin\theta\cos\theta\sin^2\omega - 2\,r\sin\theta\cos\theta\cos^2\omega)\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t} = -2\,r\sin\theta\cos\theta\,(1)\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}.$

$(r\sin^2\theta\sin\omega\cos\omega - r\sin^2\theta\sin\omega\cos\omega)\frac{d\,\Omega}{d\,r} = 0\frac{d\,\Omega}{d\,r} = 0.$ We have also the corresponding term $0\,\frac{1}{k}\frac{d\,P}{d\,r} = 0.$

$(\sin\theta\cos\theta\sin\omega\cos\omega - \sin\theta\cos\theta\sin\omega\cos\omega)\frac{d\,\Omega}{d\,\theta} = 0\frac{d\,\Omega}{d\,\theta} = 0.$ We have also the corresponding term $0\,\frac{1}{k}\frac{d\,P}{d\,\theta} = 0.$

$\left(\frac{-r\sin\theta\sin^2\omega}{r\sin\theta}-\frac{r\sin\theta\cos^2\omega}{r\sin\theta}\right)\frac{d\,\Omega}{d\,\phi}=-1\frac{d\,\Omega}{d\,\phi}$. We have also the corresponding term $-\frac{1}{k}\frac{d\,P}{d\,\phi}$.

Taking the sum of these results and placing it $=0$, we have,

(7³) $$\frac{1}{k}\frac{d\,P}{d\,\phi}=-r^2\sin^2\theta\frac{d^2\,\phi}{d\,t^2}-2\,r\,\sin^2\theta\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t}-2\,r\sin\,\theta\cos\,\theta\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}-\frac{d\,\Omega}{d\,\phi}.$$

(7¹) $$\frac{1}{k}\frac{d\,P}{d\,r}=-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t}+r\,n^2\sin^2\theta-\frac{d\,\Omega}{d\,r},$$

(7) (7²) $$\frac{1}{k}\frac{d\,P}{d\,\theta}=-r^2\frac{d^2\,\theta}{d\,t^2}-2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}+r^2\sin\,\theta\cos\,\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t}+r^2\,n^2\sin\,\theta\,\cos\,\theta-\frac{d\,\Omega}{d\,\theta},$$

(7³) $$\frac{1}{k}\frac{d\,P}{d\,\varphi}=-r^2\sin^2\theta\frac{d^2\,\varphi}{d\,t^2}-2\,r\,\sin^2\theta\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t}-2\,r^2\sin\,\theta\,\cos\,\theta\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}-\frac{d\,\Omega}{d\,\varphi}.$$

In these equations $\frac{d\,P}{d\,r}$, $\frac{1}{r}\frac{d\,P}{d\,\theta}$, and $\frac{1}{r\sin\,\theta}\frac{d\,P}{d\,\varphi}$ represent the forces arising from the pressure in the reverse directions, respectively, of r, θ, and φ.

$\frac{d\,P}{d\,r}=$ lineal force in direction r.

Let $\mu=$ lineal distance. $\mu=r\,\theta$. Radius times angular distance. $d\,\mu=r\,d\,\theta$.

$\frac{d\,P}{r\,d\,\theta}=\frac{d\,P}{d\,\mu}$ lineal.

Again,

Let $\mu=r\sin\,\theta\,\phi=$ the lineal distance at polar distance θ.

$d\,\mu=r\,\sin\,\theta\,d\,\phi$.

$\frac{d\,P}{r\sin\,\theta\,d\,\phi}=\frac{d\,P}{d\,\mu}$ lineal.

4. If N be the normal distance to the surface of the earth, or to any level surface, and the forces in the first two of the preceding equations be resolved in the directions of the normal and a perpendicular to the normal in the plane of the meridian, putting $\cos\frac{r}{N}=1$, and neglecting the small terms multiplied by $\sin\frac{r}{N}$, which are of the second order of the earth's ellipticity, and letting $\frac{1}{r}\frac{d\,P}{d\,\theta}$ represent the force arising from the pressure, resolved in the direction of the perpendicular to the normal in the plane of the meridian, the preceding equations give, when the fluid is at rest,

(8¹) $$\frac{1}{k}\frac{d\,P}{d\,_N}=r\,n^2\sin\,\theta-\frac{d\,\Omega}{d\,_N}=-g,$$

(8) (8²) $$\frac{1}{k}\frac{d\,P}{d\,\theta'}=r^2\,n^2\sin\,\theta\cos\,\theta-\frac{d\,\Omega}{d\,\theta'}=0,$$

(8³) $$\frac{1}{k}\frac{d\,P}{d\,\varphi}=-\frac{d\,\Omega}{d\,\varphi}=0,$$

and hence, neglecting the very small terms multiplied by $\sin\frac{r}{N}$, depending upon the motions of the fluid relative to the earth's surface, they give for the fluid in motion,

(9¹) $$\frac{1}{k}\frac{d\,P}{d\,_N}=-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\,\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t}-g,$$

(9) (9²) $$\frac{1}{k}\frac{d\,P}{d\,\theta'}=-r^2\frac{d^2\,\theta}{d\,t^2}-2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}+r^2\sin\,\theta\cos\,\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t},$$

(9³) $$\frac{1}{k}\frac{d\,P}{d\,\varphi}=-r^2\sin^2\theta\frac{d^2\,\varphi}{d\,t^2}-2\,r\sin^2\theta\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t}-2\,r^2\sin\,\theta\cos\,\theta\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}.$$

Integrating, we obtain

$$P=H-\int_N g\,k=H-g\,k\,N+\int_N N\frac{d\,(g\,k)}{d\,_N},$$

(10) $$=H-g\,k\,N+\int_N g\,N\frac{d\,k}{d\,_N}+\int_N N\,k\frac{d\,g}{d\,_N},$$

$$=H+K,$$

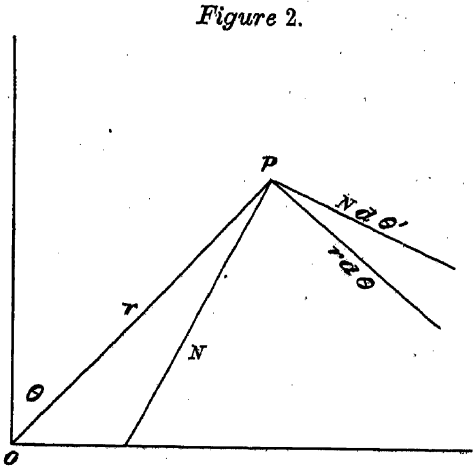
in which H must satisfy the following equations of partial differentials:

$$\frac{1}{k}\frac{d\,H}{d\,N}=-\frac{d^2\,r}{d\,t^2}+r\,\left(\frac{d\,\theta}{d\,t}\right)^2+r\,\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t},$$

$$(11)\;\frac{1}{k}\frac{d\,H}{d\,\theta'}=-\,r^2\frac{d^2\,\theta}{d\,t^2}-2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}+r^2\sin\theta\cos\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\varphi}{d\,t},$$

$$\frac{1}{k}\frac{d\,H}{d\,\varphi}=-\,r^2\sin^2\theta\frac{d^2\,\varphi}{d\,t^2}-2\,r\,\sin^2\theta\frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t}$$

Figure 2.



$$-\,2\,r^2\sin\theta\cos\theta\frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t},$$

and in which

$$(12)\quad K=-\,g\,k\,N+\int_N g\,N\frac{d\,k}{d\,N}+\int_N N\,k\frac{d\,g}{d\,N}.$$

Hence H is the pressure arising from the motions of the fluid, and K that arising from its gravity. If g and k are functions of N, θ', and φ, K is a function of the same.

Equations (7) are reduced to the form (8) by the following process: Let P be the point; 0 the centre of the earth; r the radius; N the normal at the point P; $r\,d\,\theta$ the perpendicular to r at P, and $N\,d\,\theta'$ the perpendicular to N at P. The forces then are to be resolved in the direction N and $N\,d\,\theta'$.

We have then,

$$(a)\;\frac{1}{k}\frac{d\,P}{d\,N}=\left(-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}+r\,n^2\sin^2\theta-\frac{d\,\Omega}{d\,t}\right)\cos\widehat{{}^r_N}$$
$$+\left(-r^2\frac{d^2\,\theta}{d\,t^2}-2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}+r^2\sin\theta\cos\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}+r^2\,n^2\sin\theta\cos\theta-\frac{d\,\Omega}{d\,\theta}\right)\sin\widehat{{}^r_N}.$$

When the fluid is at rest we would have,

$$\frac{1}{k}\frac{d\,P}{d\,N}=\left(r\,n^2\sin^2\theta-\frac{d\,\Omega}{d\,r}\right)\cos\widehat{{}^r_N}+\left(-\frac{d\,\Omega}{d\,\theta}\right)\sin\widehat{{}^r_N},\;\text{but}\;-\left(\frac{d\,\Omega}{d\,r}\right)\cos\widehat{{}^r_N}+\left(-\frac{d\,\Omega}{d\,\theta}\right)\sin\widehat{{}^r_N}$$
$$=-\frac{d\,\Omega}{d\,r}\frac{d\,r}{d\,N}-\frac{1}{r}\frac{d\,\Omega}{d\,\theta}\frac{r\,d\,\theta}{d\,N}=-\frac{d\,\Omega}{d\,N}.$$

$$(8^1)\;\therefore\;\frac{1}{k}\frac{d\,P}{d\,N}=\left(r\,n^2\sin^2\theta-\frac{d\,\Omega}{d\,N}\right)=-\,g\quad(g\text{ being gravity.})$$

Similarly we have,

$$\frac{1}{k}\frac{d\,P}{d\,\theta'}=\left(-r^2\frac{d^2\,\theta}{d\,t^2}-2\,r\frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}+r^2\sin\theta\cos\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}+r^2\,n^2\sin\theta\cos\theta-\frac{d\,\Omega}{d\,\theta}\right)\cos\widehat{{}^r_N}$$
$$-\left(-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}+r\,n^2\sin^2\theta-\frac{d\,\Omega}{d\,r}\right)\sin\widehat{{}^r_N}$$

When the fluid is at rest,

$$\frac{1}{k}\frac{d\,P}{d\,\theta'}=\left(+r^2\,n^2\sin\theta\cos\theta-\frac{d\,\Omega}{d\,\theta}\cos\widehat{{}^r_N}-\frac{d\,\Omega}{d\,r}\sin\widehat{{}^r_N}\right).$$

$$\text{But}\;-\frac{d\,\Omega}{d\,\theta}\frac{d\,\theta}{d\,\theta'}-\frac{1}{r}\frac{d\,\Omega}{d\,r}\frac{r\,d\,r}{d\,\theta'}=-\frac{d\,\Omega}{d\,\theta'},$$

$$(8^2)\;\therefore\;\frac{1}{k}\frac{d\,P}{d\,\theta'}=\left(r^2\,n^2\sin\theta\cos\theta-\frac{d\,\Omega}{d\,\theta'}\right)=0.$$

In (7^3) all the terms except the last reduce to 0 when the fluid is at rest $\therefore$ we have,

$$(8^3)\quad\frac{1}{k}\frac{d\,P}{d\,\phi}=-\frac{d\,\Omega}{d\,\phi}.$$

All the terms except $\frac{d\,\Omega}{d\,\theta}$ in the second part of the second member of (a) reduce to 0 when we neglect the small terms multiplied by $\sin\widehat{{}^r_N}$, then we have (9^1). Similarly we have (9^2) and (9^3).

Integrating we obtain from (9^1)

$$P=\int k\left(-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}\right)d\,N-\int g\,k\,d\,N.$$

$$\text{Let}\;H=\int k\left(-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}\right)d\,N$$

Then,

$$\frac{1}{k}\frac{d\,H}{d\,N}=-\frac{d^2\,r}{d\,t^2}+r\left(\frac{d\,\theta}{d\,t}\right)^2+r\sin^2\theta\left(n+\frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}.$$

$$\text{We have then,}\;P=H-\int g\,k\,d\,N=H-g\,k\,N+\int\left(N\frac{d\,(g\,k)}{d\,N}\right)d\,N$$
$$=H-g\,k\,N+\int\left(g\,N\frac{d\,k}{d\,N}\right)d\,N+\int\left(N\,k\frac{d\,g}{d\,N}\right)d\,N$$

Let $K = -g\,k\,N + \int \left(g\,N \frac{d\,k}{d\,N}\right) d\,N + \int \left(N\,k \frac{d\,g}{d\,N}\right) d\,N.$

Then $P = H + K.$

Similarly we integrate (9^2).

$$P = \int k \left(-r^2 \frac{d^2\,\theta}{d\,t^2} - 2\,r \frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} + r^2 \sin\theta \cos\theta \left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}\right) d\,\theta'.$$

Let $H = \int k \left(-r^2 \frac{d^2\,\theta}{d\,t^2} - 2\,r \frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} + r^2 \sin\theta \cos\theta \left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}\right) d\,\theta'.$

Then, $P = H$. K being 0 in this case we may put $P = H + K$.

Also we have by integrating (9^3),

$$P = \int k \left(-r^2 \sin^2\theta \frac{d^2\,\phi}{d\,t^2} - 2\,r \sin^2\theta \frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} - 2\,r^2 \sin\theta \cos\theta \frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}\right) d\,\phi.$$

Let $H = \int k \left(-r^2 \sin^2\theta \frac{d^2\,\phi}{d\,t^2} - 2\,r \sin^2\theta \frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} - 2\,r^2 \sin\theta \cos\theta \frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}\right) d\,\phi.$

Then $P = H$. K being 0, in this case we may put $P = H + K$.

Remembering that H is the pressure arising from the motions of the fluid, and K that arising from gravity, K must of course, be a function of g and k.

5. For a stratum of equal pressure, P is constant, and hence

$$\frac{d\,P}{d\,\theta'} = 0, \qquad \frac{d\,P}{d\,\varphi} = 0.$$

If we therefore put K' and h for the special values, respectively, of K and N, belonging to a stratum of equal pressure, and take the derivatives of (10) with regard to θ' and φ, and neglect the very small terms containing $\frac{d\,r}{d\,t}$ as a factor, which, in all ordinary motions of the fluid, will be shown to be insensible, we obtain for the general equations of horizontal motions, by restoring the value of ω in § (2).

(13)
$$0 = \frac{1}{k}\frac{d\,K'}{d\,\theta'} - r^2 \frac{d^2\,\theta}{d\,t^2} + r^2 \sin\theta \cos\theta \left(2n + \frac{d\,\varphi}{d\,t}\right)\frac{d\,\varphi}{d\,t},$$
$$0 = \frac{1}{k}\frac{d\,K'}{d\,\varphi} - r^2 \sin^2\theta \frac{d^2\,\varphi}{d\,t^2} - 2\,r^2 \sin\theta\cos\theta \left(n + \frac{d\,\varphi}{d\,t}\right)\frac{d\,\theta}{d\,t},$$

in which

(14) $$K' = -g\,k\,h + \int_h g\,h \frac{d\,k}{d\,h} + \int_h h\,k \frac{d\,g}{d\,h}.$$

(10) $P = H + K.$ Let $K' = K.$ Then,

$P = H + K'.$ Dividing through by k, we get,

$\frac{1}{k} P = \frac{H}{k} + \frac{K'}{k}.$ Differentiating with respect to θ', and regarding P as constant, we obtain,

$0 = \frac{1}{k}\frac{d\,H}{d\,\theta'} + \frac{1}{k}\frac{d\,K'}{d\,\theta'}.$ Substituting the value of $\frac{1}{k}\frac{d\,H}{d\,\theta'}$, from (11) we get, where $\frac{d\,r}{d\,t} = 0$,

$0 = \frac{1}{k}\frac{d\,K'}{d\,\theta'} - r^2 \frac{d^2\,\theta}{d\,t^2} + r^2 \sin\theta \cos\theta \left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}$, but remembering from § 2 that $n\,t + \phi = \omega$, we substitute instead of $\frac{d\,\omega}{d\,t}$ its value $n + \frac{d\,\phi}{d\,t}$, and we have

(13^1) $$0 = \frac{1}{k}\frac{d\,K'}{d\,\theta'} - r^2 \frac{d^2\,\theta}{d\,t^2} + r^2 \sin\theta \cos\theta \left(2\,n + \frac{d\,\phi}{d\,t}\right)\frac{d\,\phi}{d\,t}.$$

Again, we have, $\frac{1}{k} P = \frac{H}{k} + \frac{K'}{k}.$ Differentiating with regard to ϕ we obtain in a manner similar to the above,

$0 = \frac{1}{k}\frac{d\,K'}{d\,\varphi} - r^2 \sin^2\theta \frac{d^2\,\phi}{d\,t^2} - 2\,r \sin^2\theta \frac{d\,\omega}{d\,t}\frac{d\,r}{d\,t} - 2\,r^2 \sin\theta \cos\theta \frac{d\,\omega}{d\,t}\frac{d\,\theta}{d\,t}.$ This becomes (13^2) where $\frac{d\,r}{d\,t} = 0$, and $n\,t + \phi = \omega$. In which we have, having substituted K' for K, and h for N in (12),

(14) $$K' = -g\,k\,h + \int g\,h \frac{d\,k}{d\,h} d\,h + \int h\,k \frac{d\,g}{d\,h} d\,h.$$

6. If we suppose the fluid to be elastic, and the ratio of the density to the elastic force or pressure to depend upon the temperature, we may put

(15) $\quad k = \alpha\,P,$

in which α may be a function of h, θ', and φ. Substituting this value of k in (14), we get, when g may be regarded as constant, since in that case the last term of (14) vanishes.

(16) $$\frac{dK'}{d\theta'} = -\frac{d\,(g\,\alpha\,P\,h)}{d\,\theta'} + \frac{d\int_h g\,h\,\frac{d\,(\alpha\,P)}{d\,h}}{d\,\theta'},$$

$$= -g\,\alpha\,P\frac{d\,h}{d\,\theta'} - g\,h\,P\,\frac{d\,\alpha}{d\,\theta'} + g\,P\frac{d\int_h h\,\frac{d\,\alpha}{d\,h}}{d\,\theta'},$$

Hence, dividing by $k = \alpha\,P$, we get

$$\frac{1}{k}\frac{d\,K'}{d\,\theta'} = -\frac{g\,d\,h}{d\,\theta'} - g\,h\,\frac{\frac{d\,a}{a}}{d\,\theta'} + \frac{g}{a}\,\frac{d\int h\left(\frac{d\,a}{d\,h}\right)d\,h}{d\,\theta'}.$$

By changing the θ' to ϕ, for the last equation, we obtain $\frac{1}{k}\frac{d\,K'}{d\,\phi} = -\frac{g\,d\,h}{d\,\phi} - \frac{g\,h\,\frac{d\,a}{a}}{d\,\phi} + \frac{g}{a}\,\frac{d\int h\left(\frac{d\,a}{d\,h}\right)d\,h}{d\,\phi}$.

Remembering that $\frac{d\,a}{a} = d\log a$ the last equations become

$$\frac{1}{k}\,\frac{d\,K'}{d\,\theta'} = -g\frac{d\,h}{d\,\theta'} - g\,h\,\frac{d\log\alpha}{d\,\theta'} + A_{\theta'},$$

(17) $$\frac{1}{k}\,\frac{d\,K'}{d\,\varphi} = -g\frac{d\,h}{d\,\varphi} - g\,h\,\frac{d\log\alpha}{d\,\varphi} + A_{\phi},$$

by changing θ' to φ for the last equation, and putting

$$A_{\theta'} = \frac{g}{\alpha}\,\frac{d\int_h h\,\frac{d\,\alpha}{d\,h}}{d\,\theta'},$$

(18) $$A_{\phi} = \frac{g}{\alpha}\,\frac{d\int_h h\,\frac{d\,\alpha}{d\,h}}{d\,\varphi}.$$

7. When $\frac{d\,\alpha}{d\,h}$ is constant, that is, when α varies as the altitude,

We have,

$$A_{\theta'} = \frac{g}{a}\,\frac{d\left(\frac{1}{2}h^2\frac{d\,a}{d\,h}\right)}{d\,\theta'} = \frac{g\,\frac{d\,a}{d\,h}\,d\,h^2}{2\,a\,d\,\theta'}$$

$$A_{\phi} = \frac{g}{a}\,\frac{d\left(\frac{1}{2}h^2\frac{d\,a}{d\,h}\right)}{d\,\phi} = \frac{g\,\frac{d\,a}{d\,h}\,d\,h^2}{2\,a\,d\,\phi}$$ and these become

$$A_{\theta'} = \frac{g\,e}{2\,\alpha}\,\frac{d\,h^2}{d\,\theta'},$$

(19) $$A_{\phi} = \frac{g\,e}{2\,\alpha}\,\frac{d\,h^2}{d\,\varphi},$$

in which

$$e = \frac{d\,\alpha}{d\,h},$$

depending upon the constant ratio of increase or decrease of α with h.

8. When α is constant, or when the fluid is homogeneous, the last two terms of (17) vanish, and, in the latter case, h may represent the height of the surface of the fluid above any level surface.

9. In the preceding investigation, the effect upon g arising from a change of the figure of the fluid has been neglected. It is small in the case of water, and in the case of the atmosphere, entirely insensible.

10. Equations (13), together with the condition that the same amount of fluid must always occupy a space which is inversely as its density, the analytical expression of which is called the equation of continuity, are the conditions which must be satisfied by the motions of a fluid surrounding the earth, aud are sufficient to determine its horizontal motions, and also the value of h, which gives the figure of the fluid. When h is determined, equation (10) gives the pressure of the fluid. As only a very special form of the general equation of continuity will be needed in this investigation, it is unnecessary to give it here.

SECTION II.

ON THE MOTIONS AND FIGURE OF A FLUID SURROUNDING THE EARTH.

11. The results obtained in this and the following section, will be upon the hypothesis that the motions of the fluid are not resisted by the earth's surface. The motions and figure of the fluid must be such as to satisfy equations (13), and also the condition of continuity. To determine them in the general case in which a and g are functions of h, θ', and φ, would be very difficult. We shall take here the special case only in which a is a function of θ' which increases or decreases, from the equator to the pole, and in which g is regarded as constant at all parts of the earth's surface, and throughout the whole range of altitude. Equations (13) become in this case, when the fluid is elastic,

or when $a = \frac{k}{P}$ is variable and the value of $\frac{1}{k}\frac{d K'}{d \theta'}$ and $\frac{1}{k}\frac{d K'}{d \phi}$ as found in (17) being also substituted for these values in (13), and $\frac{d a}{d h}$ being constant,

$$g \frac{d h}{d \theta'} = r^2 \sin \theta \cos \theta \left(2 n + \frac{d \varphi}{d t}\right) \frac{d \varphi}{d t} - r^2 \frac{d^2 \theta}{d t^2} - g h \frac{d \log a}{d \theta'}$$

$$(20) \quad g \frac{d h}{d \varphi} = - 2 r^2 \sin \theta \cos \theta \left(n + \frac{d \varphi}{d t}\right) \frac{d \theta}{d t} - r^2 \sin^2 \theta \frac{d^2 \varphi}{d t^2}.$$

In inelastic fluids we have $\log k$ instead of $\log a$.

$k = a P.$
$\operatorname{Log} k = \log a + \log P.$
$d \log k = d \log a.$ P being a constant.

12. To determine completely the motions and figure of the fluid which would satisfy the conditions for any initial state of the fluid, upon the hypothesis that the motions are entirely free from resistances arising from the motion of the particles amongst themselves, would be impossible. But since in all fluids there are slight resistances to the motions of the particles amongst themselves, which, however small, eventually destroy all oscillatory or wave motions depending upon the initial state of the fluid, and reduce the motions of the particles amongst themselves to the minimum which satisfies the conditions, it is not necessary to integrate the equations generally, but merely to satisfy them with the least possible motion of the particles amongst themselves. Hence both the motions and density of the fluid at any place, and likewise its figure, must be independent of the time, and therefore constant.

13. Since $\frac{d h}{d \varphi}$ in the last of equations (20) can only have a value arising from an oscillatory or wave motion of the fluid, which would soon be destroyed by friction, it must be put equal 0, and then the equation gives by integration for each particle supposed to be entirely free from the resistances arising from the motions of the particles amongst themselves,

$$(21) \qquad r^2 \sin^2 \theta \left(n + \frac{d \varphi}{d t}\right) = c,$$

in which c is a constant depending upon the initial motion of the particle. Let
R be the radius of the earth regarded as constant,
m the mass of fluid, and
l its uniform depth when at rest relative to the earth.

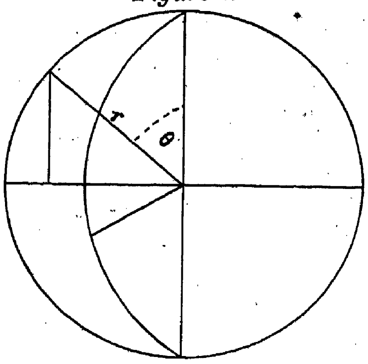
As the quantity of motion in the whole mass cannot be affected by the mutual actions of the particles upon each other, we have, even in the case in which the particles are not free from mutual resistances,

$$(22) \qquad \textstyle\int_m r^2 \sin^2 \theta \left(n + \frac{d \varphi}{d t}\right) = \int_m c = C m,$$

in which C is a constant depending upon the initial motions of all the particles.

The first member of this equation expresses the half sum of the areas projected upon the plane of the equator, arising from the absolute motions of all the particles for a unit of time, and hence this sum is constant.

Figure 3.



$$0 = \int - 2\, r^2 \sin\theta \cos\theta \left(n + \frac{d\phi}{dt}\right) \frac{d\theta}{dt} - \int r^2 \sin^2\theta \frac{d^2\phi}{dt^2}.$$

$$\text{Or, } 0 = - r^2 \sin^2\theta \left(n + \frac{d\phi}{dt}\right) + c.$$

(21) Or, $r^2 \sin^2\theta \left(n + \frac{d\phi}{dt}\right) = c.$ Equation of conservation of areas. C is the mean of all the c's, c being the areal velocity.

$r \sin\theta =$ projection at the plane of the equator in the same meridian.

$(r\sin\theta)\left(n + \frac{d\phi}{dt}\right) =$ the distance between the two radii on the equator. Then the whole double area of the triangle on the equator is,

$$(r\sin\theta)(r\sin\theta)\left(n + \frac{d\phi}{dt}\right) = r^2\sin^2\theta\left(n + \frac{d\phi}{dt}\right).$$

14. If we put v for $\frac{d\varphi}{dt}$ belonging to the initial state of the fluid, the last equation gives, neglecting quantities of the order of the range of altitude compared with the earth's radius,

$$\begin{aligned} C\,m &= R^2 \int_m \sin^2\theta\,(n+v), \\ &= R^4 \int_0^l{}_N \int_0^{2\pi}{}_\phi \int_0^\pi{}_\theta\, k \sin^3\theta\,(n+v), \\ &= \tfrac{2}{3} R^2\, m\,(n+v') \end{aligned}$$

in which

$$v' = \frac{3}{2}\frac{R^2}{m} \int_0^l{}_N \int_0^{2\pi}{}_\phi \int_0^\pi{}_\theta\, k\, v \sin^3\theta.$$

Hence,

$$C = \tfrac{2}{3} R^2 (n + v').$$

In the preceding integration k is supposed to be independent of θ. If the density should vary considerably with the latitude, it would affect the preceding result slightly.

When, by the mutual actions of the different strata upon one another, $\frac{d\varphi}{dt}$ becomes the same at all altitudes, upon the same parallel of latitude, c then becomes equal to C, and equation (21) gives

(23) $$\frac{d\varphi}{dt} = \frac{C}{R^2\sin^2\theta} - n = \frac{2\,(n+v')}{3\sin^2\theta} - n.$$

This value of $\frac{d\varphi}{dt}$ satisfies the last of equations (20), and since it gives a uniform motion of all the particles of the fluid upon the same parallel of latitude, as much fluid flows from any place as flows into it, while the destiny, from what has been stated, remains the same, and hence it also satisfies the condition of continuity.

Remembering that $R = r$, we have,

$C\,m = R^2 \int \sin^2\theta\,(n+v)\,dm.$ We have $dm = k\,dr,\ r\,d\theta,\ r\sin\theta\,d\phi = k\,r^2\sin\theta\,dr\,d\theta\,d\phi.$

Substituting this value of dm in the equation $C\,m = R^2 \int \sin^2\theta\,(n+v)\,dm$ and we have,

$C\,m = R^2 \int \sin^2\theta\,(n+v)\,(k\,r^2 \sin\theta\,dr\,d\theta\,d\phi).$

Or, $C\,m = R^4 \int \sin^3\theta\,k\,(n+v)\,dr\,d\theta\,d\phi.$

$C\,m = R^4 \int_0^l dN \int_0^{2\pi} d\phi \int_0^\pi \theta\,k \sin^3\theta\,(n+v)$ where $r = N$.

The limits of N being 0 and l, the height of the liquid.

The limits of ϕ being 0 and 2π, the circumference of the sphere, or 360° of longitude.

The limits of θ being 0 and π, or the whole amount of north polar distance that is possible.

$$m = \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\,R^2 \sin\theta.$$

$$R^4 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta\,(n+v) = \tfrac{2}{3} R^2 \left(\tfrac{3}{2} R^2 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta\,n + \tfrac{3}{2} R^2 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta\,v\right).$$

(a) $$= \tfrac{2}{3} R^2\, \tfrac{3}{2} R^2 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta \left(n + \frac{\tfrac{3}{2} R^2 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta\,v}{\tfrac{3}{2} R^2 \int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta}\right)$$

but $\int_0^l dr \int_0^{2\pi} d\phi \int_0^\pi d\theta\,k\sin^3\theta = 2\pi \int_0^l dr \int_0^\pi d\theta\,k\sin^3\theta.$

$\text{Sin}^3\,\theta\, d\,\theta = ((\sin^2\theta)\sin\theta)\,d\,\theta = ((1-\cos^2\theta)\sin\theta)\,d\,\theta = \sin\theta\, d\,\theta - \cos^2\theta\sin\theta\, d\,\theta.$

Integrating this we have, $-\cos\theta + \frac{1}{3}\cos^3\theta = -\cos\theta + \frac{1}{3}(1-\sin^2\theta)\cos\theta = -\cos\theta + \frac{\cos\theta}{3} - \frac{1}{3}\sin^2\theta\cos\theta$

$= -\frac{1}{3}\sin^2\theta\cos\theta - \frac{2}{3}\cos\theta.$

We have, then, taking the definite integral,

$$2\pi\int_0^l d\,r\,k\left(-\tfrac{1}{3}\sin^2\theta\cos\theta - \tfrac{2}{3}\cos\theta\right)_0^\pi$$

$$= 2\pi\int_0^l d\,r\,k\left(-\tfrac{1}{3}(\cos\theta\cos^3\theta) - \tfrac{2}{3}\cos\theta\right)_0^\pi = 2\pi\int_0^l d\,r\,k\left(\left(\left(-\tfrac{1}{3}(-1+1) - \tfrac{2}{3}(-1)\right) - \left(\tfrac{1}{3}(1-1) - \tfrac{2}{3}(1)\right)\right)\right)$$

$$= 2\pi\int_0^l d\,r\,k\left(+\tfrac{2}{3}+\tfrac{2}{3}\right) = 2\pi\int_0^l d\,r\,k\,\tfrac{4}{3} = \tfrac{8}{3}\pi\int_0^l d\,r\,k.$$

Again,

$$m = \int_0^l d\,r\int_0^{2\pi} d\,\phi\int_0^\pi d\,\theta\, k\, R^2\sin\theta = \int_0^l d\,r\,k\,R^2\,\phi_0^{2\pi}\,(-\cos\theta)_0^\pi = \int_0^l d\,r\,k\,R^2\,2\,\pi\,(+1+1) = \int_0^l d\,r\,k\,R^2\,4\,\pi.$$

Or, $\int_0^l d\,r = \frac{m}{4\,\pi\,k\,R^2}.$

Substituting this value of $\int_0^l d\,r$ in the value $\frac{8}{3}\pi\int_0^l d\,r\,k$, given above, we have,

$$\int_0^l d\,r\int_0^{2\pi} d\,\phi\int_0^\pi d\,\theta\,k\sin^3\theta = \frac{8}{3}\,\frac{\pi\,m}{4\,\pi\,k\,R^2}\,k = \frac{2}{3}\,\frac{m}{R^2}.$$

This we can substitute in (a) and we have the member,

$$\tfrac{2}{3}R^2\left(n + \frac{\frac{3}{2}R^2\int_0^l d\,N\int_0^{2\pi} d\,\phi\int_0^\pi d\,\theta\,\sin^3\theta\,k\,v}{m}\right).$$

Let $\frac{3}{2}\,\frac{R^2}{m}\int_0^l d\,N\int_0^{2\pi} d\,\phi\int_0^\pi d\,\theta\,k\,v\sin^3\theta = v'$. Then, $C\,m = \frac{2}{3}R^2\,m\,(n+v')$. Hence, dividing by m, we have,

$C = \frac{2}{3}R^2\,(n+v').$

15. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, equation (23) becomes

Equation (23) becomes, as $v' = 0$,

$\frac{d\,\phi}{d\,t} = \frac{2\,n}{3\sin^2\theta} - n.$ Or,

$$\frac{d\,\varphi}{d\,t} = \left(\frac{2}{3\sin^2\theta} - 1\right)n, \tag{24}$$

substituting this value of $\frac{d\,\varphi}{d\,t}$ in the first of equations (20) we get, by putting R for r,

$$g\,\frac{d\,h}{d\theta'} = R^2\sin\theta\cos\theta\left(2\,n\left(\frac{2}{3\sin^2\theta}-1\right)n + \left(\frac{2}{3\sin^2\theta}-1\right)^2 n^2\right) - R^2\,\frac{d^2\,\theta}{d\,t^2} - g\,h\,\frac{d\log a}{d\,\theta'}.$$

Or,

$$g\,\frac{d\,h}{d\theta'} = R^2\,n^2\sin\theta\cos\theta\left(\frac{4}{3\sin^2\theta} - 2 + \frac{4}{9\sin^4\theta} - \frac{4}{3\sin^2\theta} + 1\right) - R^2\,\frac{d^2\,\theta}{d\,t^2} - g\,h\,\frac{d\log a}{d\,\theta'}.$$

Or,

$$g\,\frac{d\,h}{d\,\theta'} = R^2\,n^2\sin\theta\cos\theta\left(\frac{4}{9\sin^4\theta} - 1\right) - R^2\,\frac{d^2\,\theta}{d\,t^2} - g\,h\,\frac{d\log a}{d\,\theta'}. \tag{25}$$

The last term of this equation is a function of h, the height of the strata, and the equation can only be satisfied by counter-currents of the strata between the equator and the poles; for, as we have seen that the figure of the fluid, and its density at the same place, are constant, in order to satisfy the condition of continuity, these currents must be such as to satisfy, for every vertical column of the fluid, the following equation,

$$\int_m \frac{d\,\theta'}{d\,t} = 0. \tag{26}$$

In order to satisfy this condition, h, which is a general integral, must have a negative constant added to it. Hence at a certain altitude the last term of (20) vanishes, and the fluid there has no motion towards or from the equator.

Adding a constant of integration c, to the general integral (14), by which is meant an integral without the constant of integration determined, we get,

$$K' = -g\,k\,h + \int g\,h\,\frac{d\,k}{d\,h}\,d\,h + c,$$

in which c being a constant of integration with regard to the variable N in (10) may be a function of (θ). With the constant supplied we get for (16), in the special case which we are considering here, § 11, in which case the last term of the second form of the expression in (16) vanishes,

$$\frac{d k'}{d \theta'} = - g\, a\, P \frac{d h}{d \theta'} - g\, h\, P \frac{d a}{d \theta'} + \frac{d c}{d \theta}.$$

Dividing this by $k = a\, P$ we get,

$$\frac{1}{k} \frac{d K'}{d \theta'} = - g \frac{d h}{d \theta'} - g\, h \frac{d \log a}{d \theta'} + \frac{1}{k} \frac{d c}{d \theta} = - g \frac{d h}{d \theta'} - g\,(h - h') \frac{d \log a}{d \theta'}$$

in which $h' = \dfrac{d c}{d \theta} \dfrac{1}{g\, k \dfrac{d \log a}{d \theta'}}.$

h' is the altitude where $\frac{d h}{d \theta}$ in (25), the gradient of the stratum of equal pressure, is unaffected by the temperature of which a is a function, and hence when $h = h'$, that is at the altitude of h' the last term of (25), which contains the disturbing force vanishes, and consequently $\frac{d^2 \theta}{d t^2}$, and there is no motion at that altitude to or from the poles, and when there is no east or west motion $\frac{d h}{d \theta'}$ also vanishes. The value of h' must be such that there is as much motion toward the pole in the upper strata where h is greater than h', and consequently the last term in (25) is negative, as there is motion toward the pole in the lower strata where h is less than h', and where consequently the last term in (25) is positive, and consequently gives a contrary motion.

If the problem could be completely solved in any case, the value of h' would be determined by (26), but generally we cannot get quantatitive results, and can only infer that h in the last term of (25) must have a negative constant, which shows that the stratum of no motion between the equator and the pole is at some elevation above the earth's surface. [The AUTHOR.]

If the density increases towards the poles, this term is positive for the lower strata, but negative for the upper ones, and hence the motion is toward the equator below, and from it above. If the density decreases toward the poles, the motions are the reverse.

If there were no resistances of any kind, the motions would be continually accelerated so long as the density is different between the equator and the poles; but where there are slight resistances, the motions are only accelerated until the resistances become equal to the accelerating force.

16. Since the last two terms of (25) have a very little effect upon the value of h, and consequently upon the figure of the fluid, in comparison with the remaining term of that member, unless the difference of density, and the motion of the fluid between the equator and the poles, are very great, we shall neglect them, and determine the figure depending upon the remaining term arising from the earth's rotation. Equation (25) gives by integration in this case, since $\frac{d h}{d \theta'}$ does not differ sensibly from $\frac{d h}{d \theta}$,

$$2\, g\, h = - R^2 n^2 \left(\frac{4}{9 \sin^2 \theta} + \sin^2 \theta \right) + C.$$

If h' be put for the value of h at the equator, putting $\sin \theta = 1$, we get

$$2\, g\, h' = - \tfrac{13}{9} R^2 n^2 + C.$$

Hence,

(27)
$$h = h' + \frac{R^2 n^2}{2 g} \left(\tfrac{13}{9} - \frac{4}{9 \sin^2 \theta} - \sin^2 \theta \right).$$

Since one of the terms in his value of h has $\sin \theta$ in the denominator, whatever be the value of h at the equator, it must become 0 towards the poles, and the surface of the fluid meet the surface of the earth; and this must be the case, however large the terms which have been neglected. Hence *the fluid, however deep it may be at the equator, cannot exist near the poles.*

$$g\, h = \int R^2 n^2 \sin \theta \cos \theta \left(\frac{4}{9 \sin^4 \theta} - 1 \right) d \theta' = R^2 n^2 \int \left(\frac{4 \cos \theta\, d \theta}{9 \sin^3 \theta} - \sin \theta \cos \theta\, d \theta \right)$$

$$= R^2 n^2 \int \left(\frac{4}{9} (\sin \theta)^{-3} \cos \theta\, d \theta) - (\sin \theta \cos \theta\, d \theta) \right) = R^2 n^2 \left(\frac{4}{9} \left(- \tfrac{1}{2} (\sin \theta)^{-2} \right) - \tfrac{1}{2} \sin^2 \theta) \right) + C$$

$$= - \tfrac{1}{2} R^2 n^2 \left(\frac{4}{9 \sin^2 \theta} + \sin^2 \theta \right) + C.$$

$$2\, g\, h = - R^2 n^2 \left(\frac{4}{9 \sin^2 \theta} + \sin^2 \theta \right) + C.$$

We have, $2\, g\, h' = - R^2 n^2 \left(\frac{4}{9} + \frac{9}{9} \right) + C.$

$$2\,g\,h' = -\frac{13}{9}R^2 n^2 + C.$$

Hence, as $h = -\frac{R^2 n^2}{2g}\left(\frac{4}{9\sin^2\theta} + \sin^2\theta\right) + C$, and $h' = -\frac{R^2 n^2}{2g}\left(\frac{13}{9}\right) + C$, we have,

$$h - h' = \frac{R^2 n^2}{2g}\left(\frac{13}{9} - \frac{4}{9\sin^2\theta} - \sin^2\theta\right),$$ or transposing the h' we get (27).

17. If θ_0 be the value of θ where $h = 0$, the last equation gives

(28) $$\sin^4\theta_0 - \left(\frac{2g}{R^2 n^2}h' + \tfrac{13}{9}\right)\sin^2\theta_0 = -\tfrac{4}{9},$$

which determines θ_0 when h' is given.

If we put θ_1 for the value of θ where h is a maximum, equation (25), putting $\frac{d\,h}{d\,\theta'} = 0$, and neglecting the last two terms, gives

(29) $$\sin^2\theta_1 = \tfrac{2}{3},$$

This gives $\theta_1 = 55°$ nearly, answering to the parallel of 35°, where h is a maximum.

If, therefore, we assume h', equation (27) gives the figure which the fluid assumes, *which must be somewhat as represented in the external part of* Fig. (1), *the surface of the fluid being slightly depressed at the equator, having its maximum height about the parallel of* 35°, *and meeting the surface of the earth towards the pole.*

We have, $h\sin^2\theta_0 = h'\sin^2\theta_0 + \frac{R^2 n^2}{2g}\frac{13}{9}\sin^2\theta_0 - \frac{R^2 n^2}{2g}\frac{4}{9} - \frac{R^2 n^2}{2g}\sin^4\theta_0$, and dividing through by $\frac{R^2 n^2}{2g}$ we get, after transposing and combining terms, the equation (28).

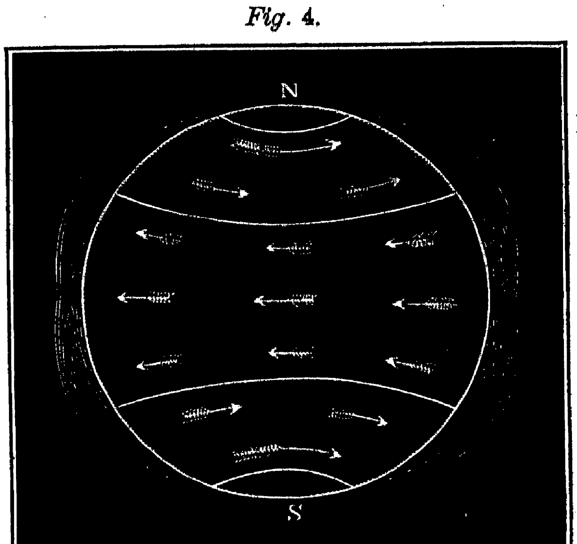
$\frac{R^2 n^2}{g}\sin\theta_{\prime}\cos\theta_{\prime}\left(\frac{4}{9\sin^4\theta} - 1\right) = 0.\quad \frac{4}{9\sin^4\theta_{\prime}} = 1.\quad \frac{4}{9} = \sin^4\theta_{\prime}.$ Extracting the square root of both sides we have,

(29) $\quad \text{Sin}^2\theta_{\prime} = .66666 \therefore 2\log\sin\theta_{\prime} = \log .6666.\quad \text{Log}\sin\theta_{\prime} = 9.91195 \therefore \theta_{\prime} = 54°\,44'.$

18. In the applications of the preceding equations we must put

$$R = 3{,}956 \text{ miles} = 20{,}887{,}680 \text{ feet},$$

Fig. 4.



$$n = \frac{2\pi}{(23 \times 60 + 56)\,60} = .000072924$$

$$g = 32.2 \text{ feet}.$$

Hence $\quad R\,n = 1{,}523.2$ feet, and $\frac{R^2 n^2}{2g} = 36{,}017$ feet.

With these values, if we assume $h' = 5$ miles, (28) gives $\theta_0 = 28°\,30'$ for the polar distance of the parallel where the surface of the fluid, or the stratum of equal pressure, meets the surface of the earth.

If in (27) we substitute for $\sin\theta$ its special value in (29), we obtain $h - h' = 4{,}002$ feet for the excess of the height of the fluid at its maximum, above its height at the equator; which is a constant independent of the amount or depth of the fluid.

19. If we put $\frac{d\,\varphi}{d\,t} = 0$ in (24), it gives

$2n - n\,3\sin^2\theta = 0.$

$\frac{2}{3} = \sin^2\theta.$ But by (29) $\sin^2\theta_{\prime} = \frac{2}{3} \therefore$ (30)

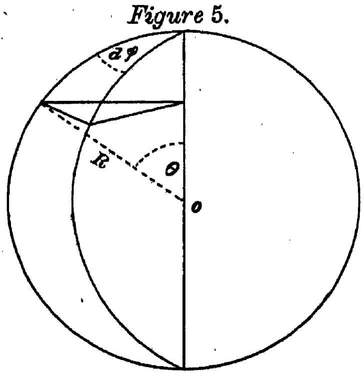
(30) $$\sin^2\theta = \sin^2\theta_1 = \tfrac{2}{3}.$$

Hence, *the latitude of no motion of the fluid east or west, is the latitude of the maximum of h.*

20. If $\sin^2\theta < \frac{2}{3}$ in (24), $\frac{d\,\varphi}{d\,t}$ is positive, but if $\sin^2\theta > \frac{2}{3}$, it is negative. Hence, *between the parallels of* 35° *and the poles, the motion of the fluid is eastward, but between those parallels and the equator it is toward the west.*

21. The lineal velocity of the fluid east or west, relative to the earth's surface is $R\sin\theta\frac{d\,\varphi}{d\,t}$.

Figure 5.



This is shown by the aid of the diagram.

$R \sin \theta =$ radius of the circular section, perpendicular to the axis, at the point on the earth's surface.

Representing it by v'', equation (24) gives

$$v'' = R \sin \theta \frac{d\phi}{dt} = R \sin \theta \left(\frac{2}{3 \sin^2 \theta} - 1\right) n$$

(31) $$v'' = R\, n \left(\frac{2}{3 \sin \theta} - \sin \theta\right).$$

Putting $\sin \theta = 1$, this equation gives $v'' = -\frac{1}{3} R\, n = 508$ feet for the velocity of the fluid westward per second at the equator. Toward the poles it is evident, from an inspection of the preceding equation, that the velocity must become very great.

This is true, because if we decrease the value of the $\sin \theta$ the value of v'' is increased.

The east and west motions of the fluid, as well as its figure, are represented by Fig. (4), the different lengths of arrows representing, in some measure, the different velocities of the fluid.

22. The whole lineal velocity of the fluid east, arising from both the earth's rotation and the velocity of the fluid relative to the earth's surface, is $R \sin \theta \left(n + \frac{d\varphi}{dt}\right)$. Representing this by v''', equation (24) gives

$$v''' = R \sin \theta \left(n + \frac{d\phi}{dt}\right)$$

$$v''' = R \sin \theta \left(n + \frac{2n}{3 \sin^2 \theta} - n\right) = \frac{2 R \sin \theta\, n}{3 \sin^2 \theta} = \frac{2 R n}{3 \sin \theta}$$

(32) $$v''' = \frac{2 R n}{3 \sin \theta}.$$

Hence this velocity is inversely as the distance from the axis of rotation,

because $R \sin \theta =$ distance from the axis of rotation, whose value depends on variations in $\sin \theta$ alone.

$\therefore \frac{R}{\sin \theta}$ would vary inversely as the distance from the axis of rotation,

which is a necessary consequence of the preservation of areas, as shown in § 13. For as the fluid in moving from the equator toward the poles approaches the axis of rotation, it must have its velocity increased, and in receding from the axis it must be decreased, just as a planet is accelerated in its perihelion but retarded in its aphelion. The reasoning, therefore, of those who, in attempting to explain the trade-winds, assume that the fluid, in moving towards or from the equator, has a tendency to retain the same lineal velocity, is erroneous.

23. If, instead of a state of rest relative to the earth's surface, we suppose that the fluid has an initial angular velocity, we must put $n + v'$ instead of n in the preceding equations.

SECTION III.

ON THE MOTIONS AND FIGURE OF A SMALL CIRCULAR PORTION OF FLUID ON THE EARTH'S SURFACE.

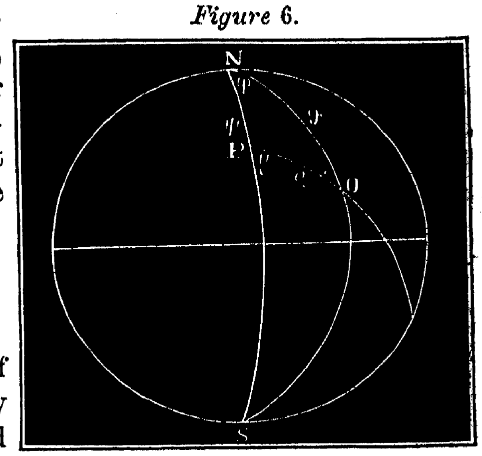
24. We shall, in this case, suppose that a is a function of the distance from the center of the fluid. It will be more convenient, therefore, to express our general equations (20) in terms of other polar co-ordinates, of which the pole P, Fig. 6, does not correspond with the pole of the earth. Regarding the earth as a perfect sphere, let ψ be the distance in arc of the new pole P from the pole of the earth; also let

ρ be the distance in arc from the pole P,

μ the angle $S\,P\,O$ between ρ and the meridian,

ε the alternate angle $N\,O\,P$.

Figure 6.



If in equations (20) we put $n = 0$, they become the equations of horizontal motions in the case in which the earth has no rotary motion, and the pole of the co-ordinates can in this case be assumed at pleasure. Hence, when the earth has no rotation, by putting ρ for θ, and μ for φ, we have

$$g\frac{d\,h}{d\,\rho}=r^2\sin\rho\cos\rho\left(\frac{d\,\mu}{d\,t}\right)^2-r^2\frac{d^2\,\rho}{d\,t^2}-g\,h\,\frac{d\log\alpha}{d\,\rho},$$

(33) $$g\frac{d\,h}{d\,\mu}=-\,2\,r^2\sin\rho\cos\rho\,\frac{d\,\rho}{d\,t}\,\frac{d\,\mu}{d\,t}-r^2\sin^2\rho\,\frac{d^2\,\mu}{d\,t^2}.$$

When the earth has a rotation, we must add to the second members of the equations respectively the terms $\frac{d\,F}{d\,\rho}$, and $\frac{d\,F}{d\,\mu}$, in which F is the part of P, equation (10) depending upon the earth's rotation, and must satisfy the following equations,

$$\frac{d\,F}{d\,\theta}=2\,r^2\,n\sin\theta\cos\theta\,\frac{d\,\varphi}{d\,t}.$$

$$\frac{d\,F}{d\,\varphi}=-\,2\,r^2\,n\,\sin\theta\,\cos\theta\,\frac{d\,\theta}{d\,t}.$$

Since θ and φ are functions of ρ and μ, we must put

$$\frac{d\,F}{d\,\rho}=\frac{d\,F}{d\,\theta}.\,\frac{d\,\theta}{d\,\rho}+\frac{d\,F}{d\,\varphi}.\,\frac{d\,\varphi}{d\,\rho},$$

$$\frac{d\,F}{d\,\mu}=\frac{d\,F}{d\,\theta}.\,\frac{d\,F}{d\,\mu}+\frac{d\,F}{d\,\varphi}.\,\frac{d\,\varphi}{d\,\mu}.$$

Hence, substituting the preceding values of $\frac{d\,F}{d\,\theta}$ and $\frac{d\,F}{d\,\varphi}$, we get

$$\frac{d\,F}{d\,\rho}=2\,r^2\,n\,\sin\theta\,\cos\theta\left(\frac{d\,\varphi}{d\,t}\,\frac{d\,\theta}{d\,\rho}-\frac{d\,\theta}{d\,t}.\,\frac{d\,\varphi}{d\,\rho}\right),$$

(34) $$\frac{d\,F}{d\,\mu}=2\,r^2\,n\,\sin\theta\,\cos\theta\left(\frac{d\,\varphi}{d\,t}\,\frac{d\,\theta}{d\,\mu}-\frac{d\,\theta}{d\,t}.\,\frac{d\,\varphi}{d\,\mu}\right).$$

Now, from the relations of the different parts of a spherical triangle, we have

$$\cos\theta=\cos\psi\cos\rho+\sin\psi\sin\rho\cos(180^\circ-\mu),$$

$$\text{and }\cot\phi=\frac{\cot\rho\sin\psi-\cos\psi\cos(180^\circ-\mu)}{\sin\mu}.$$

Taking $\cos(180^\circ-\mu)=-\cos\mu$ and multiplying and dividing the second member of the second equation by $\sin\rho$ we get,

$$\cos\theta=\cos\psi\cos\rho-\sin\psi\sin\rho\cos\mu,$$

(35) $$\cos\varphi=\frac{\sin\psi\cos\rho+\cos\psi\sin\rho\cos\mu}{\sin\rho\sin\mu}.$$

Hence, taking the derivatives and reducing, we get

$$-\sin\theta\,\frac{d\,\theta}{d\,\rho}=-\cos\psi\sin\rho-\sin\psi\cos\rho\cos\mu,$$

$$\text{or, }\frac{d\,\theta}{d\,\rho}=\frac{\cos\psi\sin\rho+\sin\psi\cos\rho\cos\mu}{\sin\theta}.$$

But, according to Chauvenet's Trig., eq. (7), page 154, $\cos\varepsilon\sin\theta=\cos\psi\sin\rho-\sin\psi\cos\rho\cos(180^\circ-\mu)$,

$$\text{or, }\cos\varepsilon=\frac{\cos\psi\sin\rho+\sin\psi\cos\rho\cos\mu}{\sin\theta}.$$

$$\therefore\frac{d\,\theta}{d\,\rho}=\frac{\cos\psi\sin\rho+\sin\psi\cos\rho\cos\mu}{\sin\theta}=\cos\varepsilon,$$

also, $\frac{d\,\theta}{d\,\mu}=-\frac{\sin\psi\sin\rho\sin\mu}{\sin\theta}$. But we have, $\frac{\sin(180^\circ-\mu)}{\sin\theta}=\frac{\sin\varepsilon}{\sin\psi}$, or $\sin\varepsilon=\frac{\sin\mu\sin\psi}{\sin\theta}$,

$$\frac{d\,\theta}{d\,\mu}=-\frac{\sin\psi\sin\rho\sin\mu}{\sin\theta}=-\sin\rho\sin\varepsilon,$$

$$\text{Also, }-(1+\cot^2\phi)\,\frac{d\,\phi}{d\,\rho}=\frac{\sin\mu\sin\rho\sin\psi(-\sin\rho)+\sin\mu\sin\rho\cos\psi\cos\mu\cos\rho}{\sin^2\psi\sin^2\mu}$$

$$-\frac{\sin\psi\cos\rho\sin\mu\cos\rho+\cos\psi\sin\rho\cos\mu\sin\mu\cos\rho}{\sin^2\psi\sin^2\mu}$$

$$=-\frac{\sin\mu\sin\psi(\sin^2\rho+\cos^2\rho)}{\sin^2\mu\sin^2\rho}=-\frac{\sin\psi}{\sin\mu\sin^2\rho}\;\therefore\;\frac{d\,\phi}{d\,\rho}=\frac{\sin^2\phi\sin\psi}{\sin\mu\sin^2\rho}.$$

$\frac{\sin^2\phi\sin\psi}{\sin\mu\sin^2\rho}=\frac{\sin^2\phi}{\sin^2\mu\sin^2\rho}\,\frac{\sin\mu\sin\psi\sin\theta}{\sin\theta}=\frac{\sin^2\phi\sin\theta\sin\varepsilon}{\sin^2\rho\sin^2\mu}=\frac{\sin^2\phi\sin^2\theta\sin\varepsilon}{\sin^2\rho\sin^2\mu\sin\theta}=\frac{\sin\varepsilon}{\sin\theta}$, because the remainder of the last term reduces to unity.

$$\therefore \frac{d\,\varphi}{d\,\rho} = \frac{\sin^2\varphi \sin\psi}{\sin\mu \sin^2\rho} = \frac{\sin\varepsilon}{\sin\theta},$$

Also, $\dfrac{d\,\phi}{d\,\mu} = (-\sin^2\phi)\,\dfrac{-\sin\mu\sin^2\rho\cos\psi\sin\mu - \sin\psi\cos\rho\sin\rho\cos\mu - \cos\psi\sin^2\rho\cos^2\mu}{\sin^2\mu\sin^2\rho},$

$\dfrac{d\,\phi}{d\,\mu} = \sin^2\phi\,\dfrac{(\sin^2\rho\cos\psi(\sin^2\mu + \cos^2\mu) + \sin\psi\cos\rho\sin\rho\cos\mu)}{\sin^2\mu\sin^2\rho},$

$\dfrac{d\,\phi}{d\,\mu} = \sin^2\phi\,\dfrac{(\cos\psi\sin\rho + \sin\psi\cos\rho\cos\mu)}{\sin^2\mu\sin\rho}.$

Substituting $\cos\varepsilon$ in this equation for its value found before we have,

$\sin^2\phi\left(\dfrac{\cos\varepsilon\sin\theta}{\sin^2\mu\sin\rho}\right) = \left(\dfrac{\sin\phi}{\sin\rho}\,\dfrac{\sin\theta}{\sin\mu}\right)\dfrac{\cos\varepsilon\sin\phi}{\sin\mu}.$

The quantity $\dfrac{\sin\phi}{\sin\rho}\,\dfrac{\sin\theta}{\sin\mu}$ reduces to unity. $\therefore$ we have, $\dfrac{d\,\phi}{d\,\mu} = \dfrac{\cos\varepsilon\sin\phi}{\sin\mu}$. Multiply by $\dfrac{\sin\rho}{\sin\phi}$ and we have, $\dfrac{\sin\rho\cos\varepsilon}{\sin\mu}$ and dividing by $\dfrac{\sin\theta}{\sin\mu}$, because $\dfrac{\sin\rho}{\sin\phi} = \dfrac{\sin\theta}{\sin\mu}$, and we have, $\dfrac{d\,\phi}{d\,\mu} = \dfrac{\sin\rho\cos\varepsilon}{\sin\theta}$ $\therefore$

$$\frac{d\,\varphi}{d\,\mu} = \frac{\cos\psi\sin\rho + \sin\psi\cos\rho\cos\mu}{\sin^2\mu\sin\rho}\sin^2\varphi = \frac{\sin\rho\cos\varepsilon}{\sin\theta},$$

$$\frac{d\,\theta}{d\,t} = \frac{d\,\theta}{d\,\rho}.\,\frac{d\,\rho}{d\,t} + \frac{d\,\theta}{d\,\mu}.\,\frac{d\,\mu}{d\,t} = \cos\varepsilon\,\frac{d\,\rho}{d\,t} - \sin\rho\sin\varepsilon\,\frac{d\,\mu}{d\,t},$$

$$\frac{d\,\varphi}{d\,t} = \frac{d\,\varphi}{d\,\rho}.\,\frac{d\,\rho}{d\,t} + \frac{d\,\varphi}{d\,\mu}.\,\frac{d\,\mu}{d\,t} = \frac{\sin\varepsilon}{\sin\theta}\frac{d\,\rho}{d\,t} + \frac{\sin\rho\cos\varepsilon}{\sin\theta}\frac{d\,\mu}{d\,t}.$$

These values being substituted in (34), we get

$$\frac{d\,F}{d\rho} = 2\,r^2\,n\sin\theta\cos\theta\left(\left(\frac{\sin\varepsilon}{\sin\theta}\frac{d\,\rho}{d\,t} + \frac{\sin\rho\cos\varepsilon}{\sin\theta}\frac{d\,\mu}{d\,t}\right)\cos\varepsilon - \left(\cos\varepsilon\,\frac{d\,\rho}{d\,t} - \sin\rho\sin\varepsilon\,\frac{d\,\mu}{d\,t}\right)\frac{\sin\varepsilon}{\sin\theta}\right),$$

$$\frac{d\,F}{d\,\rho} = 2\,r^2\,n\cos\theta\,(\cos^2\varepsilon + \sin^2\varepsilon)\sin\rho\,\frac{d\,\mu}{d\,t},$$

(36[1]) $\dfrac{d\,F}{d\,\rho} = 2\,r^2\,n\cos\rho\,\cos\theta\,\dfrac{d\,\mu}{d\,t}.$

Also, $\dfrac{d\,F}{d\,\mu} = 2\,r^2\,n\sin\theta\cos\theta\left(\left(\dfrac{\sin\varepsilon}{\sin\theta}\dfrac{d\,\rho}{d\,t} + \dfrac{\sin\rho\cos\varepsilon}{\sin\theta}\dfrac{d\,\mu}{d\,t}\right)(-\sin\rho\sin\varepsilon) - \left(\cos\varepsilon\,\dfrac{d\,\rho}{d\,t} - \sin\rho\sin\varepsilon\,\dfrac{d\,\mu}{d\,t}\right)\dfrac{\sin\rho\cos\varepsilon}{\sin\theta}\right),$

$\dfrac{d\,F}{d\,\mu} = +\,2\,r^2\,n\cos\theta\,(-(\sin^2\varepsilon + \cos^2\varepsilon)\sin\rho)\,\dfrac{d\,\rho}{d\,t},$

(36[2]) $\dfrac{d\,F}{d\,\mu} = -\,2\,r^2\,n\sin\rho\cos\theta\,\dfrac{d\,\rho}{d\,t}.$

(36)
$$\frac{d\,F}{d\,\rho} = 2\,r^2\,n\,\sin\rho\,\cos\theta\,\frac{d\,\mu}{d\,t},$$
$$\frac{d\,F}{d\,\mu} = -\,2\,r^2\,n\,\sin\rho\cos\theta\,\frac{d\,\rho}{d\,t}.$$

If we add these values of $\dfrac{d\,F}{d\,\rho}$ and $\dfrac{d\,F}{d\,\mu}$ respectively to the second members of (33), we get for the equations of motion, in terms of ρ and μ, when the earth has a rotation,

(37)
$$g\,\frac{d\,h}{d\,\rho} = r^2\sin\rho\left(2\,n\cos\theta + \frac{d\,\mu}{d\,t}\cos\theta\right)\frac{d\,\mu}{d\,t} - r^2\frac{d^2\,\rho}{d\,t^2} - g\,h\,\frac{d\log\alpha}{d\,\rho},$$
$$g\,\frac{d\,h}{d\,\mu} = -\,2\,r^2\,\sin\rho\left(n\cos\theta + \frac{d\,\mu}{d\,t}\cos\rho\right)\frac{d\,\rho}{d\,t} - r^2\sin^2\rho\,\frac{d^2\,\mu}{d\,t^2},$$

in which $\cos\theta$ has the value in terms of ρ and μ, in the first of (35).

25. When $\sin\rho$ is so small that the last term of the value of $\cos\theta$ may be neglected in comparison with the first, we have $\cos\theta = \cos\psi\cos\rho$, which being substituted in the last equations, they become

(38)
$$g\,\frac{d\,h}{d\,\rho} = r^2\,\sin\rho\,\cos\rho\left(2\,n\cos\psi + \frac{d\,\mu}{d\,t}\right)\frac{d\,\mu}{d\,t} - r^2\frac{d^2\,\rho}{d\,t^2} - g\,h\,\frac{d\log\alpha}{d\,\rho},$$
$$g\,\frac{d\,h}{d\,\mu} = -\,2\,r^2\sin\rho\cos\rho\left(n\cos\psi + \frac{d\,\mu}{d\,t}\right)\frac{d\,\rho}{d\,t} - r^2\sin^2\rho\,\frac{d^2\,\mu}{d\,t^2}.$$

These equations are similar to equations (20)

(20).
$$\begin{cases} g\,\dfrac{d\,h}{d\,\theta'} = r^2\sin\theta\,\cos\theta\left(2\,n + \dfrac{d\,\phi}{d\,t}\right)\dfrac{d\,\phi}{d\,t} - r^2\,\dfrac{d^2\,\theta}{d\,t^2} - g\,h\,\dfrac{d\log\alpha}{d\,\theta'} \\ g\,\dfrac{d\,h}{d\,\phi} = -\,2\,r^2\sin\theta\cos\theta\left(n + \dfrac{d\,\phi}{d\,t}\right)\dfrac{d\,\theta}{d\,t} - r^2\sin^2\theta\,\dfrac{d^2\,\phi}{d\,t^2} \end{cases}$$

having ρ and μ instead of θ and φ, and, instead of n, having $n \cos \psi$, which is the earth's angular velocity of rotation around the axis, corresponding with the pole P (Peirce's "Analytical Mechanics," § 25). Hence we can treat these equations precisely as equations (20) in the last section, and, instead of (21), we get

as $\frac{d h}{d \mu}$ must be equal to 0, because the motion will be destroyed in this direction,

$$0 = \int - 2 r^2 \sin \rho \cos \rho \left(n \cos \psi + \frac{d \mu}{d t}\right) \frac{d \rho}{d t} - \int r^2 \sin^2 \rho \frac{d^2 \mu}{d t^2},$$

$$0 = r^2 \sin^2 \rho \left(n \cos \psi + \frac{d \mu}{d t}\right) + C, \text{ or,}$$

(39) $$r^2 \sin^2 \rho \left(n \cos \psi + \frac{d \mu}{d t}\right) = c,$$

and, instead of (22), we get

and, by the same reasoning as employed there

(40) $$\int_m r^2 \sin^2 \rho \left(n \cos \psi + \frac{d \mu}{d t}\right) = \int_m c = C m.$$

On account of the term which has been neglected in the value of $\cos \theta$, these equations cannot be used for large values of ρ, and hence we may put $\sin \rho = \rho$. Let

$s = R \rho$ be the lineal distance from the centre,

s' be the value of s at the external part of the fluid,

u be the initial value of $\frac{d \mu}{d t}$.

The last equation then gives, putting R for r,

$$\begin{aligned} C m &= \int_m s^2 (n \cos \psi + u,) \\ &= \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s k s^3 (n \cos \psi + u), \\ &= \tfrac{1}{2} s'^2 m (n \cos \psi + u',) \end{aligned}$$

in which

$$u' = \frac{2}{s'^2 m} \int_0^l {}_N \int_0^{2\pi} {}_\mu \int_0^{s'} {}_s k s^3 u.$$

$$C m = \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 (n \cos \psi + u) = \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 n \cos \psi + \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u.$$

$$m = \int_0^l d N \int_0^{2\pi} s' d \mu \int_0^{s'} d s\, k$$

$$m = 2 \pi k \int_0^l d N \tfrac{1}{2} s'^2$$

$$\int_0^l d N = \frac{m}{\pi k s'^2}.$$

$$C m = 2 \pi \int_0^l d N \tfrac{1}{4} s'^4 k n \cos \psi + \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u = 2 \pi \frac{m}{4 \pi k s'^2} s'^4 k n \cos \psi + \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u$$

$$= m \cos \psi s'^2 n \cos \psi + \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u = \tfrac{1}{2} s'^2 m \left(n \cos \psi + \frac{2}{s'^2 m} \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u\right).$$

Then $C m = \frac{1}{2} s'^2 m (n \cos \psi + u')$, in which $u' = \frac{2}{s'^2 m} \int_0^l d N \int_0^{2\pi} d \mu \int_0^{s'} d s\, k s^3 u$.

Hence,

(41) $$C = \tfrac{1}{2} s'^2 (n \cos \psi + u').$$

In the preceding integration k is regarded as a constant. When, by the mutual action of the different strata upon each another, $\frac{d \mu}{d t}$ becomes the same at all altitudes at the same distance from the centre P, c becomes equal to C, and equation (39) then gives

(42) $$\frac{d \mu}{d t} = \frac{C}{R^2 \sin^2 \rho} - n \cos \psi = \frac{s'^2 (n \cos \psi + u')}{2 s^2} - n \cos \psi.$$

The last term is obtained by substituting s^2 for $R^2 \rho^2$ and $\frac{1}{2} s'^2 (n \cos \psi + u')$ for C, as is given in (41).

26. If we suppose the initial state of the fluid to be that of rest relative to the earth's surface, the last equation becomes

(43) $$\frac{d\,\mu}{d\,t}=\left(\frac{s'^2}{2\,s^2}-1\right)n\cos\psi.$$

Substituting this value of $\frac{d\,\mu}{d\,t}$ in the first of equations (38), it becomes, by putting R for r and $\cos\rho=1$

$$\frac{R^2}{R}\frac{g\,d\,h}{d\,s}=R^2\,(\rho)\,(1)\,(2\,n\cos\psi+\left(\frac{s'^2}{2\,s^2}-1\right)n\cos\psi)\left(\left(\frac{s'^2}{2\,s^2}-1\right)n\cos\psi\right)-\frac{R^2}{R}\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}\frac{R^2}{R}$$

$$g\frac{d\,h}{d\,s}=R\,(\rho)\,(n^2\cos^2\psi\frac{s'^2}{s^2}-2\,n^2\cos^2\psi+\left(\frac{s'^2}{2\,s^2}-1\right)^2n^2\cos^2\psi)-\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}$$

$$g\frac{d\,h}{d\,s}=R\,\rho\,n^2\cos^2\psi\left(\frac{s'^2}{s^2}-2+\frac{s'^4}{4\,s^4}-\frac{s'^2}{s^2}+1\right)-\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}$$

$$g\frac{d\,h}{d\,s}=R\,\rho\,n^2\cos^2\psi\left(\frac{s'^4}{4\,s^4}-1\right)-\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}$$

$\therefore$ (44) $$g\frac{d\,h}{d\,s}=n^2\cos^2\psi\left(\frac{s'^3}{4\,s^3}-s\right)-\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}$$

The value of $R\,\rho=s$ being substituted in the preceding equation.

(44) $$g\frac{d\,h}{d\,s}=n^2\cos^2\psi\left(\frac{s'^4}{4\,s^3}-s\right)-\frac{d^2\,s}{d\,t^2}-g\,h\frac{d\log a}{d\,s}.$$

This equation is similar to (25), and, like it, can only be satisfied by means of an interchanging motion between the internal and external part of the fluid; and the remarks following that equation in § 15 are also applicable to this.

27. By omitting the last two terms in the preceding equation, as was done in equation (25), (§ 16), we get by integration,

$$2\,g\,h=-\,n^2\cos^2\psi\left(\frac{s'^4}{4\,s^2}+s^2\right)+C.$$

Hence, eliminating C,

(45) $$h=h'+\frac{n^2\cos^2\psi}{2\,g}\left(\tfrac{5}{4}\,s'^2-\frac{s'^4}{4\,s^2}-s^2\right).$$

$$g\int d\,h=\int n^2\cos^2\psi\left(\frac{s'^4}{4\,s^3}-s\right)d\,s$$

$$g\,h=\int\left(n^2\cos^2\psi\left(\frac{s'^4}{4\,s^3}\,d\,s-s\,d\,s\right)\right)$$

$$g\,h=\int\left(n^2\cos^2\psi\left(\frac{s'^4}{4}\,s^{-3}\,d\,s-s\,d\,s\right)\right)$$

$$g\,h=n^2\cos^2\psi\,(s'^4\,\tfrac{1}{4}\,(-\tfrac{1}{2})\,s^{-2}-\tfrac{1}{2}\,s^2)+C$$

(*a*) $$h=\frac{-\,n^2\cos^2\psi}{2\,g}\left(\frac{s'^4}{4\,s^2}+s^2\right)+C.$$

Hence we get, by means of the following reduction, (45). Taking the value of s at the external part of the fluid we have $s=s'$, and substituting this for the last equation, taking also the new value h' for h, and we have,

$$2\,g\,h'=-\,n^2\cos^2\psi\left(\tfrac{1}{4}\,\frac{s'^4}{s'^2}+s'^2\right)+C.$$

Or (*b*). $$h'=\frac{-\,n^2\cos^2\psi}{2\,g}\,(\tfrac{5}{4}\,s'^2)+C.$$

Eliminating C from (*a*) and (*b*) we have,

$$h-h'=\frac{n^2\cos^2\psi}{2\,g}\left(\tfrac{5}{4}\,s'^2-\frac{s'^4}{4\,s^2}-s^2\right)$$

or (45).

Since one of the negative terms in this value of h has s in the denominator, it must become equal 0 towards the centre where s vanishes. Hence *the fluid, however deep it may be at the external part, cannot exist at the centre.*

28. If we put s_0 for the value of s where $h=0$, the last equation gives

(46) $$0=h'+\frac{n^2\cos^2\psi}{2\,g}\left(\tfrac{5}{4}\,s'^2-\frac{s'^4}{s_0^2}-s_0^2\right),$$

from which we obtain s_0 for any assumed value of h'.

Since s_0 is very small, the terms $\frac{5}{4}\,s'_2$ and $-\,s_0^2$ may generally be omitted in the last equation, and it then becomes

(47) $$s^0=\frac{n\cos\psi\,s'^2}{\sqrt{2\,g\,h'}}$$

by means of the following reduction:

$$0 = h' + \frac{n^2 \cos^2 \psi}{2g}\left(-\frac{s'^4}{4 s_0^2}\right).$$

$$0 = h' s_0^2 - \frac{n^2 \cos^2 \psi}{2g} s'^4. \qquad h' s_0^2 = \frac{n^2 \cos^2 \psi \, s'^4}{2g}.$$

$$s_0^2 = \frac{n^2 \cos^2 \psi \, s'^4}{2 g h'}. \qquad s_0 = \frac{n \cos \psi \, s'^2}{\sqrt{2 g h'}}.$$

If we put s_1 for s where h is a maximum, equation (44) gives, by putting $\frac{d\,h}{d\,s} = 0$, and neglecting the last two terms,

$$0 = n^2 \cos^2 \psi \left(\frac{s'^4}{4 s_/^3} - s_/\right). \qquad \frac{s'^4}{4 s_/^3} = s_/. \qquad s'^4 = 4 s_/^4. \qquad s'^2 = 2 s_/^2. \qquad s' = s_/ \sqrt{2}. \quad \text{Or (48)}$$

(48) $$s_1 = \frac{s'}{\sqrt{2}}.$$

Equation (45) determines the figure of the surface of the fluid, *which is very slightly convex towards the external part, and meets the surface of the earth near the centre c,* as represented in Figure 7.

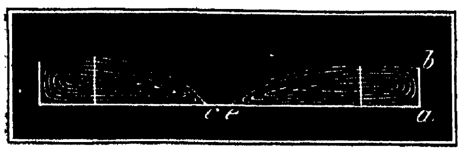


Figure 7.

If we assume h', or $a\,b$, Figure 7, equal 5 miles, and $a\,c = 100$ miles, equation (47) gives $c\,e = 2$ miles nearly.

29. Equation (43) gives the angular velocity of gyration, which must be very great near the centre, where s is small.

Putting $\frac{d\,\mu}{d\,t} = 0$, it gives

$$\left(\frac{s'^2}{2 s^2} - 1\right) n \cos \psi = 0. \qquad \text{Or, } \frac{s'^2}{2 s^2} - 1 = 0. \qquad s'^2 = 2 s^2. \qquad s = \frac{s'}{\sqrt{2}}.$$

(49) $$s = \frac{s'}{\sqrt{2}} = s_1.$$

Hence, *at the distance of* s_1, *which is the distance of the maximum of* h, *there is no gyratory motion.*

In the northern hemisphere, where $\cos \psi$ is positive, if $s < s_1$, $\frac{d\,\mu}{d\,t}$ is positive, but if $s > s_1$, it is negative. Hence *the inner part of the fluid gyrates from right to left, but the external part from left to right,* as represented in Figure 8. In the southern hemisphere, where $\cos \psi$ is negative, the gyrations are the reverse.

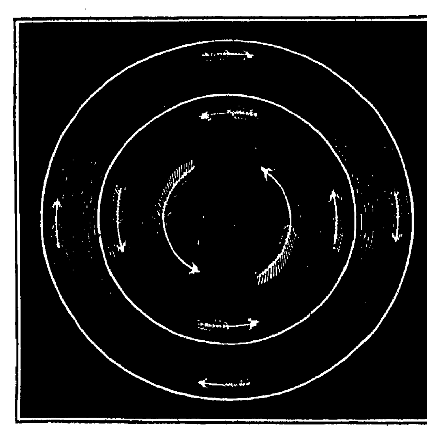


Figure 8.

30. If the fluid is of uniform density, and every part gyrates with the same angular velocity u, it satisfies equations (38) by satisfying the following equation:

$$g \frac{d\,h}{d\,s} = 2\,s\,u\,n \cos \psi + s\,u^2,$$

since all the other terms vanish; and this motion also satisfies the condition of continuity.

The last term of (38) vanishes on account of uniform density. If every particle keeps the same distance from P, then ρ is constant. If every part gyrates with the same angular velocity it must be continuous.

By integrating we get

By integrating $g \frac{d\,h}{d\,s} = 2\,s\,u\,n \cos \psi + s\,u^2$ we get,

$$\int g\,d\,h = \int (2\,s\,u\,n \cos \psi\,d\,s + s\,u^2\,d\,s)$$
$$g\,h = \tfrac{2}{2} s^2\,u\,n \cos \psi + \tfrac{1}{2} u^2 s^2 + C. \qquad \therefore (50)$$

(50) $$g\,h = \tfrac{1}{2} s^2\,u\,(2\,n \cos \psi + u) + C.$$

This is the equation of a parabola. Hence *the surface of the fluid, relative to the earth's surface, is a paraboloid.* If the portion of the fluid is so small that the earth's surface may be regarded as a plane, it becomes absolutely the surface of a paraboloid; and when the angular velocity of gyration

is great in comparison with that of the earth's rotation, $2n\cos\psi$ may be omitted, in the preceding equation, in connection with u.

The $2n\cos\psi$ comes from the earth's rotation. u is the angular velocity of rotation.

If $u = -2n\cos\psi$, or $u = 0$, h is the constant, and then the surface of the fluid is a level surface. If u is negative and less than $2n\cos\psi$, the surface is convex; in all other cases it is concave.

31. If the whole of a gyrating mass of fluid has a tendency to move in the direction of the meridian with a force V, if we regard the forces which act upon each part of the fluid in the directions of the meridians as parallel, we have, using R for r,

$$V = m\frac{d^2\psi}{dt^2} = \frac{1}{R}\int_m \frac{dP}{d\theta'}.$$

From section 3 we have $\frac{1}{R}\frac{dP}{d\theta'}$ as representing the force arising from the pressure in the direction of θ.

The error arising from regarding the forces in the directions of the meridians parallel is of the second order of their deviation from parallelism, and consequently very small, unless the lateral extent of the fluid is very great.

From the last equation and the second of equations (9), omitting the term containing $\frac{dr}{dt}$ as a factor, since it can produce no sensible effect, we get

$V = \frac{1}{R}\int dm\left(-R^2\frac{d^2\theta}{dt^2} + R^2\left(\sin\theta\cos\theta\left(n + \frac{d\omega}{dt}\right)\frac{d\phi}{dt}\right)\right)$, and remembering that $n + \frac{d\phi}{dt} = \frac{d\omega}{dt}$, we have,

$$V = \int_m\left(-R\frac{d^2\theta}{dt^2} + R\sin\theta\cos\theta\left(2n + \frac{d\varphi}{dt}\right)\frac{d\varphi}{dt}\right).$$

If in this equation we substitute for $\frac{d\varphi}{dt}$ its value in section 24, and for $\frac{d^2\theta}{dt^2}$ its value derived from that of $\frac{d\theta}{dt}$ in the same section, and also for $\cos\theta$ its value in the first of equations (35), putting $\varepsilon = \mu$, since the meridians are regarded as parallel, and omitting all terms which give 0 by integration, we get,

$$RV = -2n\sin\psi\int_m s^2\cos^2\mu\,\frac{d\mu}{dt},$$
(51)
$$= -n\sin\psi\int_m s^2\,\frac{d\mu}{dt}.$$

$\frac{d\phi}{dt} = \frac{\sin\varepsilon}{\sin\theta}\frac{d\rho}{dt} + \frac{\sin\rho\cos\varepsilon}{\sin\theta}\frac{d\mu}{dt}.$

$\frac{d\theta}{dt} = \cos\varepsilon\,\frac{d\rho}{dt} - \sin\rho\sin\varepsilon\,\frac{d\mu}{dt}.$

$\frac{d^2\theta}{dt^2} = \cos\varepsilon\frac{d^2\rho}{dt^2} - \sin\rho\sin\varepsilon\frac{d^2\mu}{dt^2} - \sin\varepsilon\frac{d\mu}{dt}\cos\rho\frac{d\rho}{dt} - \frac{d\rho}{dt}\sin\varepsilon\frac{d\varepsilon}{dt} - \sin\rho\cos\varepsilon\frac{d\varepsilon}{dt}\frac{d\mu}{dt}.$

$\cos\theta = \cos\psi\cos\rho - \sin\psi\sin\rho\cos\mu.$

Substituting these values in

$$V = \int\left(-R\frac{d^2\theta}{dt^2} + R\sin\theta\cos\theta\left(2n + \frac{d\phi}{dt}\right)\frac{d\phi}{dt}\right)dm,$$

and remembering that $\varepsilon = \mu$, we get,

$$V = +R\int\Big(\overset{o}{-\cos\mu\frac{d^2\rho}{dt^2}} + \overset{o}{\sin\rho\sin\mu\frac{d^2\mu}{dt^2}} + \overset{o}{\sin\mu\frac{d\mu}{dt}\cos\rho\frac{d\rho}{dt}} + \overset{o}{\sin\mu\frac{d\rho}{dt}\frac{d\mu}{dt}} + \overset{o}{\sin\rho\cos\mu\left(\frac{d\mu}{dt}\right)^2} +$$
$$\sin\theta(\cos\psi\cos\rho - \sin\psi\sin\rho\cos\mu)\left(2n + \frac{\sin\mu}{\sin\theta}\frac{d\rho}{dt} + \frac{\sin\rho\cos\mu}{\sin\theta}\frac{d\mu}{dt}\right)\left(\frac{\sin\mu}{\sin\theta}\frac{d\rho}{dt} + \frac{\sin\rho\cos\mu}{\sin\theta}\frac{d\mu}{dt}\right)\Big)dm.$$

The terms with o above them integrate out.

In the integration it will be remembered that

$$m = \int_0^l dN\int_0^{2\pi} s\,d\mu\int_0^{s'} ds\,k.$$

$$V = R\int\left(\sin\theta(\cos\psi\cos\rho - \sin\psi\sin\rho\cos\mu)\left(2n + \frac{\sin\mu}{\sin\theta}\frac{d\rho}{dt} + \frac{\sin\rho\cos\mu}{\sin\theta}\frac{d\mu}{dt}\right)\right.$$
$$\left.\left(\frac{\sin\mu}{\sin\theta}\frac{d\rho}{dt} + \frac{\sin\rho\cos\mu}{\sin\theta}\frac{d\mu}{dt}\right)dm.\right.$$

Putting $\cos\rho = 1$ and $\sin\rho = \rho$

$$V = R\int\left(\left(2n\cos\psi + \frac{\sin\mu\cos\psi}{\sin\theta}\frac{d\rho}{dt} + \frac{\cos\psi\,\rho\cos\mu}{\sin\theta}\frac{d\mu}{dt} - 2n\rho\sin\psi\cos\mu - \frac{{}^{*}\rho\sin\psi\sin\mu\cos\mu}{\sin\theta}\frac{d\rho}{dt} - \frac{\rho^2\sin\psi\cos^2\mu}{\sin\theta}\frac{d\mu}{dt}\right)\left(\sin\mu\frac{d\rho}{dt} + \rho\cos\mu\frac{d\mu}{dt}\right)\right)dm.$$

Multiplying by the first term of the factor and placing equal to 0 those terms which would disappear after integration, we have for the corresponding terms—

$$0 + 0 + 0 - 0 - 0 - 0.$$

Multiplying by the last term we have,

$$0 + 0 + 0 - 2n\rho^2\sin\psi\cos^2\mu\left(\frac{d\mu}{dt}\right) - 0 - 0.$$

Combining these terms we have,

$$V = R\int\left(-2n\rho^2\sin\psi\cos^2\mu\frac{d\mu}{dt}\right)dm.$$

Remembering that $s = R\rho$ we get,

$$V = R\int\left(-2n\frac{s^2}{R^2}\sin\psi\cos^2\mu\frac{d\mu}{dt}\right)dm.$$

Or,

$$RV = \int\left(-2ns^2\sin\psi\cos^2\mu\frac{d\mu}{dt}\right)dm.$$

$$RV = -2n\sin\psi\int\left(s^2\cos^2\mu\frac{d\mu}{dt}\right)dm.$$

We have from trigonometry $\cos^2\mu = \frac{1}{2} + \frac{1}{2}\cos 2\mu$. Substituting this value in the above equation, and remembering that $\frac{1}{2}\cos 2\mu$ will integrate out, we have,

$$RV = -2n\sin\psi\int\left(\tfrac{1}{2}s^2\frac{d\mu}{dt}\right)dm, \quad \text{or,}$$

$$RV = -n\sin\psi\int\left(s^2\frac{d\mu}{dt}\right)dm.$$

If $\frac{d\mu}{dt}$, the angular velocity of gyration, is positive, V is negative; but positive, if $\frac{d\mu}{dt}$ is negative. Hence *if the fluid gyrates from right to left the whole mass has a tendency to move towards the north; but if from left to right, towards the south.*

If every part of a cylindrical mass having its axis of revolution vertical has the same angular velocity of gyration as in the case of solids, calling this velocity u, the preceding equation gives for the accelerating force in the direction of the meridian,

(52)
$$\frac{V}{m} = -\frac{s'^2\,u\,n\sin\psi}{2R} = -\frac{s'^2\,u\sin\psi}{2R^2 n}\times R n^2,$$
$$= -\frac{s'^2\,u\sin\psi}{2R^2 n}\times\frac{g}{289} = -\frac{g}{578}\cdot\frac{u\sin\psi}{n}\cdot\frac{s'^2}{R^2}.$$

Calling $\frac{d\mu}{dt} = u$, and integrating, remembering that $s\sqrt{2} = s'$ (49), we get the following:

$$RV = -n\sin\psi\int\left(s^2\frac{d\mu}{at}\right)dm.$$

$$\frac{RV}{m} = \frac{-n\sin\psi\,u\int(s^2)\,dm}{m}.$$

$$\dagger\frac{RV}{m} = \frac{-n\sin\psi\,u\int\int\int s^3\,ds\,dN\,d\mu}{\int\int\int s\,ds\,dN\,d\mu}.$$

$$\frac{RV}{m} = \left(\frac{-n\sin\psi\,u\,\frac{1}{4}s^4}{\frac{1}{2}s^2}\right)_0^{s'}\left(\frac{-n\sin\psi\,u\,s^2}{2}\right)_0^{s'} = \frac{V}{m} = -\frac{s'^2\,u\,n\sin\psi}{2R} = -\frac{s'^2\,u\sin\psi}{2R^2 n}\times R n^2$$ (if we multiply and divide by Rn).

$$\frac{V}{n} = -\frac{s'\,u\sin\psi}{2Rn}\times\frac{g}{289} = -\frac{g}{578}\cdot\frac{u\sin\psi}{n}\cdot\frac{s'^2}{R^2}.$$

Where the centrifugal force at the equator, $Rn^2 = \frac{g}{289}$.

32. If a body moves in the direction of ρ or s with a velocity $v = \frac{ds}{dt}$, and p be the direction of a

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* $\mathrm{Sin}^2\mu\cos\mu = (\frac{1}{2} - \frac{1}{2}\cos 2\mu)\cos\mu = \frac{1}{2}\cos\mu - \frac{1}{4}\cos^3\mu - \frac{1}{4}\cos\mu = \frac{1}{4}\cos\mu - \frac{1}{4}\cos^3\mu.$

† $\int\int\int$ refers to limits for value of m.

perpendicular to it on the left, we obtain from the last of equations (36) for the deflecting force in the direction of p, arising from the earth's rotation,

$$\frac{dF}{dp}=\frac{\frac{dF}{d\mu}}{R\sin\rho}=-2\,R\,n\cos\theta\,\frac{d\rho}{dt},$$

(53)
$$=-2\,n\cos\theta\,\frac{ds}{dt}=-\frac{2\cos\theta\,\frac{ds}{dt}}{R\,n}\times R\,n^2,$$

$$=-\frac{2\cos\theta\,\frac{ds}{dt}}{R\,n}\times\frac{g}{289}=-\frac{2\,g\,v\cos\theta}{289\,R\,n}.$$

$dp=R\sin\rho\,d\mu.$

$\frac{dF}{dp}=\frac{dF}{R\sin\rho\,d\mu}=\frac{dF}{d\mu}\,\frac{1}{R\sin\rho}=$ (from the last of equation (36).)

$-2Rn\cos\theta\frac{d\rho}{dt}=-2\,n\cos\theta\,R\,\frac{d\rho}{dt}$, and remembering that $s=R\,\rho$, §25, we have

$-2\,n\cos\theta\,\frac{ds}{dt}$, and if this be multiplied and divided by $R\,n$ we have,

$-\frac{2\cos\theta}{R\,n}\,\frac{ds}{dt}\times R\,n^2=-\frac{2\cos\theta\frac{ds}{dt}}{R\,n}\times\frac{g}{289}$, and if we let $v=\frac{ds}{dt}$ we have,

$\frac{-2\,v\,g\cos\theta}{289\,R\,n}.$

This force is negative in the northern hemisphere, and positive in the southern. Hence *in whatever direction a body moves on the surface of the earth, there is a force arising from the earth's rotation, which deflects it to the right in the northern hemisphere, but to the left in the southern.* This result was published by me in the "Mathematical Monthly" for June, 1859, several months before the extended discussion of this subject at several successive meetings of the French Academy of Sciences, in which MM. Bertrand, Babinet, Delaunay and others took part. (See "Compt. Rendus," vol. xlix., pages 638, 658, 688, 769, 775.) This is an extension of the principle upon which the theory of the trade winds is based, and which has been heretofore supposed to be true only of bodies moving in the direction of the meridian.

SECTION IV.

ON THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

33. By the general motions of the atmosphere are meant all those motions produced by a difference of density between the equatorial and polar regions arising principally from a difference of temperature. If the motions of the atmosphere were not resisted by the earth's surface, the results of the preceeding sections could be at once applied to them without any modifications, and hence toward the poles there would be a very rapid motion eastward, and in the equatorial regions towards the west, and the atmosphere would entirely recede from the poles, and be also depressed about 4,000 feet at the equator, as has been shown in section (2). Although the preceding results, when applied to the atmosphere, are very mach modified by the resistance of the earth's surface, yet they will be of great advantage in explaining its general motions; for, as there can be no resistance until there is motion, the atmosphere must have a tendency to assume, in some measure, the same motions and figure as in the case of no resistances. Hence, towards the poles the general

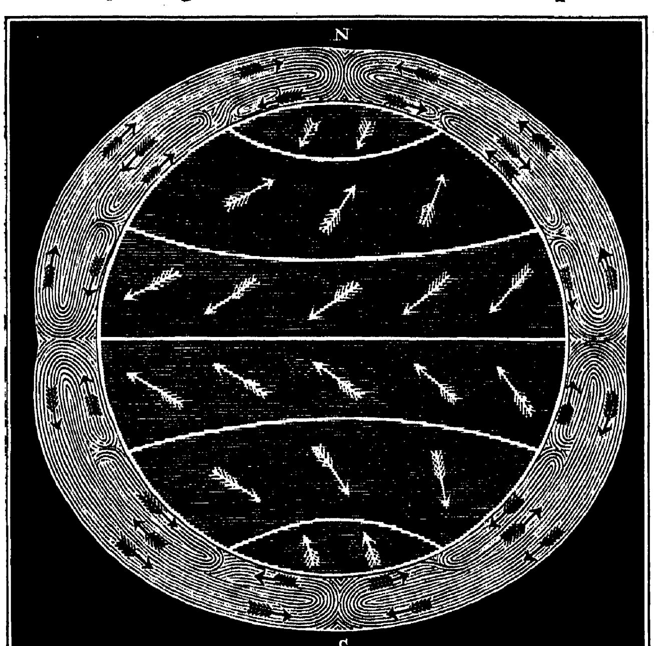

Figure 9.

motions of the atmosphere must be towards the east, and in the torrid zone towards the west, but as these motions, in consequence of the resistances, are small in comparison with those in the case of no resistances, instead of the atmosphere's receding entirely from the poles as represented in Figure 4, page 21, there must be only a comparatively small depression there, as represented in Figure 9, and instead of its being about 4,000 feet lower at the equator than at the place of its maximum height near the tropics (18), there must be only a very slight depression there.

34 The force which overcomes the resistance of the earth's surface to the east and west motions of the atmosphere depends upon the term in the least of our general equations (13) containing $\frac{d\,\theta}{d\,t}$ as a factor, which depends upon the interchanging motion of the fluid between the equatorial and the polar regions, and hence the term must vanish at the equator and the poles. All the east or west motion of the atmosphere is consequently destroyed by the resistances at these places, and hence, as $\frac{d\,\theta}{d\,t}$ vanishes there also, there is a belt of calms at the equator, called the equatorial calm-belt, and there must be also, a region of calms about the poles.

35. As the motion of the atmosphere is east towards the poles and west near the equator, somewhere between the equator and the poles there must be a parallel of no motion east or west, which, in the case of no resistance, was determined upon the hypothesis of an initial state of rest, and found to be at the parallel of 35°, (section 18.) In the case of the atmosphere, this parallel is entirely independent of the initial state of the atmosphere, and depends, in a great measure, upon the law of resistance, and hence it cannot be accurately determined. It is evident, however, that the east and west motions of the atmosphere at the earth's surface must be such that the sum of the resistances of each part of the earth's surface multiplied into its distance from the axis of rotation, must be equal 0, else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere. Now, as the part of the earth's surface where the motion of the atmosphere is west is much farther from the axis than the part where it is east, the latter part must comprise more than half of the earth's surface, unless the velocity of the eastern motion towards the poles is much greater than that of the western motion near the equator. Therefore, since one-half of the earth's surface is contained between the parallels of 30°, the parallels of no east or west motion at the earth's surface must fall within these parallels, and they are accordingly found to be near the tropics, on the ocean. Hence the maximum height of the atmosphere, as represented in Figure (9), must also be near the same parallels.

36. The increase of pressure arising from the accumulation of atmosphere near the tropics, caused, principally, by the deflecting forces (section 32) arising from the more rapid east and west motions of the atmosphere in the upper regions, where there is least resistance, gives the atmosphere a tendency to flow from beneath this accumulation both towards the equator and the poles, since the motions, and consequently the forces which cause this accumulation, are much less near the surface; but, on account of the greater density of the atmosphere towards the poles it has a tendency also to flow, at the earth's surface, from the poles towards the equator. Between the parallels of greatest pressure and the equator these tendencies combine and produce a strong surface current, which, combining with the westward motion there, gives rise to the well-known northeast wind in the northern hemisphere and the southeast wind in the southern hemisphere, called the trade-winds; but between the parallels of greatest pressure and the poles these tendencies are opposed to each other, and the one arising from the accumulation of atmosphere near the tropics, being the greater in the middle latitudes, causes the atmosphere to flow at the earth's surface towards the poles, and this motion, combining with the general eastward motion of the atmosphere in those latitudes, gives rise to the southwest wind in the northern hemisphere and the northwest wind in the southern hemisphere, called the passage-winds.

37. Near the poles the tendency to flow towards the equator seems to be greater, and causes a current there *from* the poles, which, being deflected westward (section 32), causes a slight northeast wind in the north frigid zone and a southeast wind in the south frigid zone; but this is only near the earth's surface, and the general tendency of the atmosphere in the upper regions must be towards the east, as will be seen.

38. Since the atmosphere near the tropics can have no motion in any direction at the earth's surface, there are calm-belts there, called the tropical calm-belts. Near the polar circles, where the polar and passage-winds meet, there must also be calm-belts, which may be called polar calm-belts.

The motions of the atmosphere, therefore, at the earth's surface, if they were not modified by the influence of continents, would be as represented in the interior of figure 5, in which the heavy lines represent the calm-belts. On account of the influence of the continents, these belts are somewhat displaced and irregular, and on account of the varying position of the sun, they change their positions a little in different seasons of the years. The southern limit of the polar winds in the northern hemisphere, and also the limit between the trade and passage-winds, has been determined by Professor J. H. Coffin, from the discussion of a great number of observations at different points, and given in a chart, in his treatise of the winds, published in the seventh volume of the "Smithsonian Contributions."

39. That the atmosphere is depressed at the equator and the poles, and has its maximum height near the tropics, as has been represented, is indicated by barometrical pressure. It was formerly thought that this pressure at the level of the ocean, was very nearly 30 inches in all latitudes; but it is now well established that it is much less towards the poles than near the tropics, and also a little less at the equator. Says Captain Wilkes: "The most remarkable phenomenon which our observations have shown is the irregular outline of the atmosphere surrounding the earth as indicated by the pressure upon the measured column at different parts of the surface. Our barometrical observations show a depression within the tropics, a bulging in the temperate zone, again undergoing a depression on advancing towards the arctic and antarctic circles." The mean of all the observations, as given in the Report of the Exploring Expedition, from Cape Henry to Madeira, taken between the parallels of 28° and 32°, was 31.215 inches; at Madeira, latitude 32° 53′, 30.176 inches, and in the rainy belt between the parallels of 8° and 12°, 29.987 inches. After passing the equator there was a slight elevation, again reaching its maximum near the tropic of Capricorn. Beyond this there was a gradual depression until about the parallel of 55°, where the barometer was rapidly depressed below 29 inches. After doubling Cape Horn and proceeding towards the equator, the height of the barometer gradually increased again to its usual height in the middle and equatorial latitudes. On sailing south again, in the Pacific ocean, a depression of the barometer was again observed. The mean of all the observations taken on twenty-two days, in sailing from Callao to Tahiti, between the parallels of 10° and 15°, was 30.109 inches; and of those made on thirty-two days, between the parallels of 15° and 20°, was 30.147 inches. The mean of the observations made on five days, after leaving Sydney, between the parallels of 35° and 45°, was 30.305 inches; of those made on seven days, between the parallels of 45° and 55°, was 29.790 inches: of those taken on eight days, between the parallels of 55° and 65°, was 29.378 inches. The mean also of all those taken along the antarctic continent was 29.040 inches.

40. Says Sir James Ross: ("Voyage to the Southern Seas," Vol. 2 page 383): "Our barometrical experiments appear to prove that the atmospheric pressure is considerably less at the equator than near the tropics; and to the south of the tropic of Capricorn, where it is greatest, a gradual diminution occurs as the latitude is increased, as will be shown from the following table, derived from the hourly observations of the height of the column of mercury between the 20th of November, 1839, and the 31st of July, 1843."

Extract from Ross's Table.

Latitude.	Prussure.	Latitude.	Pressure.	Latitude.	Pressure.
	Inches.		*Inches.*		*Inches.*
Equator	29.974	42° 53′	29.950	55° 52′	29.360
13° 0′ S......	30.016	45 00	29.664	60 00	29.114
22 17.........	30.085	49 08	29.467	66 00	29.078
34 48.........	30.023	51 33	29.497	74 00	28.928
		54 26	29.347		

41. The following table, first published by M. Schouw, and reduced here from millimeters to English inches, shows that there is a similar bulging of the atmosphere in the middle latitudes, and depression at the poles in the northern hemisphere, as has been observed in the southern hemisphere.

Place.	Latitude.	Pressure.	Place.	Latitude.	Pressure.
		Inches.			*Inches.*
Cape	33° 0′ S.	30.040	London	51° 30′	29.961
Rio Janeiro	23 S.	30.073	Altona	53 30	29.937
Christianburg	5 30 N.	29.925	Dantzic	54 30	29.925
La Guayra	10	29.928	Konigsburg	54 30	29.941
St. Thomas	19	29.941	Apenrade	55	29.905
Macao	23	30.039	Edinburg	56	29.851
Teneriffe	28	30.087	Christiania	60	29.866
Madeira	32 30	30.126	Bergen	60	29.703
Tripoli	33	30.213	Hardanger	60	29.700
Palermo	38	30.036	Reikiavik	64	29.607
Naples	41	30.012	Godthaab	64	29.603
Florence	43 30	29.996	Eyafiord	66	29.669
Avignon	44	30.000	Godhavn	69	29.674
Bologna	44 30	30.008	Upernavik	73	29.732
Padua	45	30.008	Mellville Isle	74 30	29.807
Paris	49	29.976	Spitzbergen	75 30	29.795

42. From the preceding tables it is seen that the barometric pressure is much less, especially in the southern hemisphere, towards the poles than at the equator, although the density towards the poles is much greater, and hence the depression there must be considerable.

43. The pressure of the atmosphere may be obtained from the first of equations (9). The terms in the equation depending upon the motions of the atmosphere are insensible, and consequently may be omitted. The term $\frac{d^2 r}{d t^2}$ depends upon the acceleration or retardation of the vertical motion of the atmosphere, and is of the same order in comparison with g as the rate of its acceleration or retardation in comparison with that of a descending or ascending free body, and hence in all ordinary motions of the atmosphere it is insensible. Restoring the value of ω in (section 2), the largest of the remaining terms is $2\, r \sin^2 \theta\, n \frac{d\,\varphi}{d\,t}$, which is of the same order in comparison with $r\, n^2$, as the east or west motion of the atmosphere in comparison with the motion of the rotation of the earth on its axis. But $r\, n^2 = \frac{1}{289}\, g$ only, hence the term is entirely insensible. We may therefore put

$$\frac{d_N\, P}{P} = -\, \alpha\, g.$$

$$(9^1). \qquad \frac{1}{k}\frac{d\,P}{d\,N} = -\frac{d^2 r}{d\,t^2} + r\left(\frac{d\,\theta}{d\,t}\right)^2 + r \sin^2\theta \left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t} - g.$$

$$\frac{1}{k}\frac{d\,P}{d\,N} = 0 + 0 + \left(2\, r\, n^2 \sin^2\theta + r \sin^2\theta\, \frac{d\,\phi}{d\,t}\right)\frac{d\,\phi}{d\,t} - g.$$

Then $\frac{1}{k}\frac{d\,P}{d\,N} = -\,g$, for the reasons given in the first part of section 43.

But $k = \alpha\, P$

$\therefore \frac{1}{k}\frac{d\,P}{d\,N} = -\,g$ can be written in the form $\frac{d\,P}{P\,d\,N} = -\,\alpha\, g$, if we substitute for k its value and multiply both members by α.

Using the common system of logarithms and putting M for its modulus, we get by integration

$$(54) \qquad \log P' - \log P = M\, \alpha\, g\, N,$$

in which P' is the pressure at the earth's surface.

Hence

Differentiating (54) with respect to θ we have

$$(55) \qquad \frac{d \log P}{d\,\theta'} = \frac{d \log P'}{d\,\theta'} - M\, g\, N \frac{d\,\alpha}{d\,\theta'}.$$

By means of this equation the second of equations (9) becomes, by putting $\alpha\, P$ for k, (section 6), and omitting the very small term $2\, r \frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t}$,

Then

$$(9^2). \qquad \frac{1}{k}\frac{d\,P}{d\,\theta'} = -\, r^2 \frac{d^2\theta}{d\,t^2} - 2\, r \frac{d\,r}{d\,t}\frac{d\,\theta}{d\,t} + r^2 \sin\theta \cos\theta \left(n + \frac{d\,\omega}{d\,t}\right)\frac{d\,\phi}{d\,t}.$$

Putting $a\,P$ for k and omitting $2\,r\,\frac{d\,r}{d\,t}\,\frac{d\,\theta}{d\,t}$ which is small we have,

$$\frac{d\,P}{P\,d\,\theta'} = a\left(r^2 \sin\theta \cos\theta\,(2\,n + \frac{d\,\phi}{d\,t})\right)\frac{d\,\phi}{d\,t} - r^2\,\frac{d^2\,\theta}{d\,t^2} \text{ or}$$

$$\frac{d \log P}{d\,\theta'} = a\left(r^2 \sin\,\theta \cos\theta\,(2\,n + \frac{d\,\phi}{d\,t})\right)\frac{d\,\phi}{d\,t} - r^2\,\frac{d^2\,\theta}{d\,t^2}.$$

Then from (55) we have,

$$(56)\qquad \frac{d \log P'}{d\,\theta'} - M\,g\,N\,\frac{d\,a}{d\,\theta'} = M.a\left(r^2 \sin\theta\cos\theta\,(2\,n + \frac{d\,\varphi}{d\,t})\frac{d\,\varphi}{d\,t} - r^2\,\frac{d^2\,\theta}{d\,t^2}\right).$$

44. In the case of the atmosphere, there must be a term in equations (9) to represent the resistances to the motions; and this term in the second of these equations may be denoted by $(\varphi)\,\frac{d\,\theta}{d\,t}$ in the second member. Putting

$$(57)\qquad W = (\varphi)\,\frac{d\,\theta}{d\,t} + r\,\frac{d^2\,\theta}{d\,t^2},$$

It will be remembered that (ϕ) means function.

W will represent the force which overcomes the resistances to the motions of the atmosphere between the equator and the poles, and also its inertia.

Since $\frac{d\,\varphi}{d\,t}$ is generally very small in comparison with $2\,n$, it may be omitted in equation (56), which then becomes, by means of equation (57),

$$(58)\qquad \frac{d \log P'}{d\,\theta'} - M\,g\,N\,\frac{d\,a}{d\,\theta'} = M\,a\,(2\,r^2\,n \sin\theta\cos\theta\,\frac{d\,\varphi}{d\,t} - r\,W).$$

45. It will be shown, that $r\,W$ is very small in comparison with $2\,r^2\,n \sin\,\theta \cos\,\theta\,\frac{d\,\varphi}{d\,t}$; and hence, if $\frac{d\,\varphi}{d\,t}$ were known, the preceding equation, neglecting the term $r\,W$, would give approximately the pressure of the atmosphere at the earth's surface. But since we do not know the value of the term denoting the resistances in the last of equations (9), we cannot determine the value of $\frac{d\,\varphi}{d\,t}$; therefore, since $\frac{d \log P'}{d\,\theta'}$ can be determined from observations of the barometric pressure, we shall use the equation to determine $\frac{d\,\varphi}{d\,t}$, from which we easily obtain the east or west motion of the atmosphere. Denoting the velocity of this motion per hour by v, we shall have

$$(59)\qquad v = 3600\,r \sin\,\theta\,\frac{d\,\varphi}{d\,t}.$$

46. The ratio of the density to the elastic force decreases $\frac{1}{491}$ for every degree of Fahrenheit. But as a higher temperature is always accompanied by a greater amount of aqueous vapor, the density of which is less than that of the atmosphere, the rate of decrease has been found to be $\frac{1}{449}$ for every degree. Let

a' be the value of a at the equator, and

i the difference of temperature between the equator and the poles.

If we suppose the temperature to decrease from the equator to the poles as the square of the sine of the latitudes, we shall have

$$a = a'\,(1 + \tfrac{1}{449}\,i \cos^2\theta).$$

Hence

$$\frac{d\,a}{d\,\theta'} = -\,\tfrac{2}{449}\,a'\,i \sin\,\theta \cos\,\theta.$$

By means of the last three equations, equation (58), putting R for r and e for $\frac{1}{M\,a'\,g}$ is reduced to

$$(60)\qquad v = \frac{1800}{R\,n \cos\theta\,(+\,\frac{1}{449}\,i \cos^2\theta)}\left(e\,g\,\frac{d \log P'}{d\,\theta'} + \tfrac{2}{449}\,i\,g \sin\,\theta \cos\theta\,N + R\,W\right).$$

$$\frac{1}{M\,a'\,g} = \varepsilon.\quad \frac{a}{1 + \frac{1}{449}\,2'\cos^2\theta} = a'.\quad \frac{1}{M\,a'} = g\,\varepsilon.\quad \text{Then, } \frac{1 + \frac{1}{449}\,2'\cos^2\theta}{M\,a} = g\,\varepsilon.\quad \text{Or, } M\,a = \frac{1 + \frac{1}{449}\,2'\cos^2\theta}{g\,\varepsilon}.$$

Since $v = 3600\, r \sin\theta \frac{d\phi}{dt}$ then, $\frac{d\phi}{dt} = \frac{v}{3600\, r \sin\theta}$.

These values being substituted in equation (58) we have,

$$\frac{d \log P'}{d\theta'} - \frac{N}{a'\varepsilon}\left(- \tfrac{2}{449}\, a'\, 2' \sin\theta \cos\theta\right) = \frac{1 + \frac{1}{449} 2' \cos^2\theta\, R}{g\,\varepsilon}\left(2\, R\, n \sin\theta \cos\theta \frac{v}{3600\, R \sin\theta} - W\right),$$

or, transposing, we have, after reducing,

$$g\,\varepsilon \frac{d \log P'}{d\theta'} \frac{1800}{R\, n \cos\theta\,(1 + \frac{1}{449} 2' \cos^2\theta)} + \tfrac{2}{449}\, 2'\, g \sin\theta \cos\theta\, N \frac{(1800)}{R\, n \cos\theta\,(1 + \frac{1}{449} 2' \cos^2\theta)} +$$
$$W\left(\frac{1 + \frac{1}{449} 2' \cos^2\theta}{g\,\varepsilon}\right)\frac{1800}{n\cos\theta} = v, \text{ or,}$$

$$(60) \qquad v = \frac{1800}{R\, n \cos\theta\,(1 + \frac{1}{449} 2' \cos^2\theta)}\left(\varepsilon\, g \frac{d \log P'}{d\theta'} + \tfrac{2}{449}\, n \sin\theta \cos\theta\, N + R\, W\right).$$

Since the variation of a with the altitude can produce no sensible effect in the results, a has been regarded as a function of the latitude only. We must, therefore, take the mean value of a' belonging to the atmosphere at the equator at all altitudes, which we will assume to be that belonging to the temperature of 32°.

47. By means of observations of P at different altitudes, equation (54) gives the value of $\frac{1}{M\,a\,g}$, which at the temperature of 32°, has been determined to be 60166 feet; which, consequently, is the value of e. The difference between the mean temperatures of the equator and the poles is about 60°; we shall, therefore, in the following applications put $i = 60$.

48. The value of $\frac{d \log P'}{d\theta'}$ in the preceding equation can be determined approximately for any latitude from the preceding tables of barometric pressure. Since the co-ordinates of pressure given there have been deduced from observations made in different longitudes and at all seasons, they are somewhat irregular; but co-ordinates can be assumed with regular differences, and such that the interpolated values of the co-ordinates of pressure for the latitudes given in the tables will very nearly correspond with the pressures given there; and then, from these co-ordinates, the approximate value of $\frac{d \log P'}{d\theta'}$ can be determined. In this manner the values of $\frac{d P}{d\theta'}$ in the following table have been determined, except the first, which has been assumed. The third column of the table contains the values of v at the earth's surface, neglecting the term W, which will be shown to have, in general, a very little effect. The fourth contains the co-efficient of N, and the fifth the value of v at the height of 3 miles.

Latitude.	$D_{\theta'} \log P'$.	v, ($N=0$).	Coeff. of N.	v, ($N=3$ miles).
75 N.	−.0060	− 2.7 miles.	2.33	+ 4.3 miles.
65	.0000	0.0	3.87	11.6
55	+.0188	+ 9.9	5.34	25.9
45	+.0080	+ 4.5	6.71	24.6
30	.0000	0.0	8.49	25.5
15	−.0060	−10.0	9.70	19.1
15 S.	+.0060	−10.0	9.70	19.1
30	−.0147	+11.4	8.49	36.9
40	−.0372	+23.4	7.36	45.5
50	−.0205	+15.3	6.07	33.5
60	−.0133	+ 0.0	4.61	19.8

In the above table the logs of the pressures, given in section 41, are so combined as to give for the different latitudes given in the first column the corresponding differences of the logarithms given in the second column. This combination is a matter of judgment, but any combination adopted would, of course, give approximately the same result. By plotting the logs of the pressure directly, it was easy to obtain, graphically, very nearly the results obtained by Professor Ferrel. Columns three, four, and five are easily computed from equation (60).

49. The term W, and its effect upon the value of v, cannot be determined, but they can be shown from observation to be, in general, very small; and, since W is positive, as may be seen from equation (57), when the motion is from the north towards the south, and negative when the contrary, except when the motion is retarded, and the term $r\frac{d^2\theta}{dt^2}$ arising from the inertia of the atmosphere is greater than the resistances, its effect for the most part is to increase the value of v algebraically where the motion is towards the south, and decrease it where it is towards the north. In the regions of the trade-winds about the parallels of 15°, the current at the surface of the earth is stronger than at any other parallel, and as the resistances at the surface must be much greater than in the upper regions,

the term W must be greater there than in any other part of the atmosphere. If $v = 0$, equation (60) gives, since $N = 0$ at the surface, $W = eg \frac{d \log P'}{d \theta'}$; and from the proceeding table, when $v = -10$ miles, $W = 0$. Now, we know from observation that the velocity of the atmosphere westward at the parallels of 15°, cannot be much less than 10 miles per hour, and hence W is small in comparison with $e g \frac{d \log P'}{d \theta'}$, which at that parallel is itself small; and hence the effect of W upon the value of v in the higher latitudes, where the value of cos θ in the denominator is much greater, must be very small. Very near the equator the formula for the value of v, equation (60), fails practically, since, on account of the small value of cos θ there, the effect of W may be very great.

50. If the motion of the atmosphere east in the higher latitudes and west near the equator, be that given in the preceding table, or by equation (60), it must cause the observed difference of barometric pressure in the different latitudes; and hence, from what we know of those motions of the atmosphere from observations, there can be no doubt that they are adequate to account for this observed difference of pressure.

51. It is evident, where the motions of the atmosphere are resisted by the earth's surface, that all the conditions cannot be satisfied by a motion at the surface from the poles towards the equator, and by a counter-motion in the upper regions. For we have seen (section 35), that the atmosphere at the surface of the earth must have an eastern motion in the middle latitudes; but it cannot have such a motion, unless it also has a motion toward the poles, in order that the deflecting force (section 18) arising from this motion may overcome the resistances to the eastern motion. But it is evident there cannot be a complete reversal of the motions in the middle latitudes, but some portion of it must flow toward the poles in the upper regions, else the eastern motion there could not be greater than at the surface, which the conditions require. The motions, therefore, must be somewhat as represented in the figure. The part of the atmosphere next the earth's surface in the middle latitudes having a motion toward the poles, extends to a considerable height, since it generally embraces the region of fair-weather clouds, as may be seen by observation.

52. It is seen, from the results given in the preceding table, that the eastward motion of the atmosphere in the middle and higher latitudes must be greatest in the upper strata, and that in the region of the trade-winds, where the motion is westward at the surface, it must be toward the east above. This is also evident from the general consideration, that the whole amount of deflecting force eastward arising from the motion of the atmosphere towards the poles is equal to the deflecting force westward arising from its motion back towards the equator, and that the deflecting force eastward is principally above where there is less resistance than near the surface. Hence at the top of Mauna Loa in the Sandwich Islands, and on the peak of Teneriffe, both of which places are near the tropical calm-belt at the surface, a strong southwest wind prevails. Hence, also, " on the eruption of Saint Vincent, in 1812, ashes were deposited at Barbadoes, sixty or seventy miles eastward, and also on the decks of vessels one hundred miles still further east, whilst the trade-wind at the surface was blowing in its usual direction." The eastward motion of the atmosphere above, in the latitudes of the trade-winds, is also confirmed by observations made on the directions of the clouds at Colonia Tovar, Venezuela, latitude 10° 26′, as given in the Report of the Smithsonian Institution for 1857 (page 254). While the motion of the lower clouds was in general from some point toward the east, the observed motion of nearly all the higher clouds was from some point toward the west.

53. From what precedes, the limit between the atmosphere which moves eastward in the middle latitudes and westward nearer the equator, which at the earth's surface is at the tropical calm-belt, must be a plane inclining toward the equator above. And since, according to section 51, the atmosphere near the earth's surface cannot have an eastward motion, unless it also has a motion toward the poles, this plane near the earth's surface must nearly coincide with the one which separates the atmosphere moving towards the poles, from that moving towards the equator, in the trade-wind regions, and hence the latter must also incline above towards the equator. This explains the winds at the peak of Teneriffe, which at the top always blow from the southwest, while at the base they blow alternately from the southwest and northeast, changing with the seasons. As the tropical calm-belt together with this dividing plane changes its position with the seasons, as will be explained, in the latter part of summer when this plane is farthest north, it still leaves the top of the peak north of it while the base is south of it; and hence the wind at the top always blows from the southwest, even when

at the base it blows from the northeast. As this plane moves south in the fall, more of the peak gradually becomes north of it, and hence the southwest wind, which always prevails at the top, gradually descends lower on the sides of the peak until it reaches the base. Hence, when this plane reaches its most southern position, in the latter part of the winter, the southwest wind prevails at both the base and the top.

54. It is seen, from the first of the results given in the last table, that if the barometric pressure increases near the poles, as it seems to do, at least in the northern hemisphere, the atmosphere at the earth's surface must have a westward motion there; and as it cannot have this motion unless it also have a motion toward the equator, so that the deflecting force arising from this motion may overcome the resistance to the westward motion, the wind there must blow slightly from the north-east, as has been shown in section 37. This, according to Professor Coffin's chart of the winds, already alluded to, seems to accord with observation.

55. The depression of the atmosphere at the poles and at the equator, and the accumulation near the tropics, may be explained in a general manner by means of the principle in (section 32) that when a body moves in any direction in the northern hemisphere, it is deflected to the right, and the contrary in the southern. The atmosphere towards the poles having an eastward motion, the deflecting force arising from it causes a pressure towards the equator, and the motion near the equator being westward, the pressure is towards the poles; and hence there must be a depression at the poles and at the equator, and an accumulation near the tropics. Since this deflecting force is as cos θ, it is small near the equator, and consequently the depression there is small.

56. According to the preceding tables of barometric pressure, there is more atmosphere in the northern than in the southern hemisphere. Says Sir James Ross, "The cause of the atmosphere being so very much less in the southern than in the northern hemisphere remains to be determined." This is very satisfactorily accounted for by the preceding principle, for, as there is much more land with high mountain ranges in the northern hemisphere than in the southern, the resistances are greater, and consequently the eastward motions, upon which the deflecting force depends, is much less; and the consequence is, that the more rapid motions of the southern hemisphere cause a greater depression there, and a greater part of the atmosphere to be thrown into the northern hemisphere.

This also accounts for the mean position of the equatorial calm-belt being, in general, a little north of the equator. But in the Pacific ocean, where there is nearly as much water north of the equator as south, its position nearly coincides with the equator.

For the same reason the tropical calm-belt of the northern hemisphere is farther from the equator than that of the southern hemisphere, and, on account of the irregular distribution of the land and water of the two hemispheres in different longitudes, it does not coincide with any parallel of latitude. In the longitude of Asia, where there is all land in the northern hemisphere and the Indian ocean in the southern, this belt, which is also the dividing line which separates the winds which blow east from those which blow west, is farther from the equator than at any other place, as shown by Professor Coffin's chart.

57. In winter, the difference of temperature between the equator and the poles, upon which the disturbance of the atmosphere depends, is much greater than in summer. This causes the eastward motion of the atmosphere in either hemisphere during its winter to be greater, while in the other hemisphere it is less. Hence, a portion of the volume of the atmosphere in winter is thrown into the other hemisphere; but, although the volume or height of the atmosphere is then less, yet, being more dense, the barometric pressure remains nearly the same. The difference at Paris, and in the middle latitudes generally, between winter and summer, is only about $\frac{1}{10}$ of an inch.

On account of this alternate change with the seasons of the velocity of the eastward motion of the atmosphere in the two hemispheres, the equatorial and tropical calm-belts change their positions a little, moving north during our spring and south in the fall.

When the sun is near the tropics, the true law of the decrease of temperature from the equator to the poles varies from that which has been assumed, (section 46), and is then different in the two hemispheres, which doubtless has some effect also upon the position of the calm-belts.

SECTION V.

ON THE MOTIONS OF THE ATMOSPHERE ARISING FROM LOCAL DISTURBANCES.

58. Besides the general disturbance of equilibrium arising from a difference of specific gravity between the equator and the poles, which causes the general motions of the atmosphere, treated in the last section, there are also more local disturbances, arising from a greater rarefaction of the atmosphere over limited portions of the earth's surface, which give rise to the various irregularities in its motions, including cyclones or revolving storms, tornadoes, and water-spouts. When, on account of greater heat, or a greater amount of aqueous vapor, the atmosphere at any place becomes more rare than the surrounding portions, it ascends, and the surrounding heavier atmosphere flows in below, to supply its place, while a counter-current is consequently produced above. As the lower strata of atmosphere generally contain a certain quantity of aqueous vapor, which is condensed after arising to a certain height, and forms clouds and rain, the caloric given out in the condensation, in accordance with Espy's theory, produces a still greater rarefaction, and doubtless adds very much to the disturbance of equilibrium, and to the motive power of storms. So long, then, as the ascending atmosphere over the area of greater rarefaction is supplied with aqueous vapor by the current flowing in from all sides below, the disturbance of equilibrium must continue, and consequently the local disturbances of the atmosphere to which it gives rise, whether those of an ordinary rain-storm, or a cyclone, may continue many days, while the general motions of the atmosphere may carry this disturbed area several thousands of miles.

In the ordinary rain-storms of the United States, the area of greater rarefaction seems to be, in general, very oblong in the direction of the meridians, as shown by Espy's charts. The atmosphere becoming more rare over the land, a current seems to set in from the Atlantic towards the Rocky mountains, causing an ascent of the atmosphere in the west, and a line of greatest rarefaction in the direction of the meridians, arising from the condensation of the ascending vapor into clouds and rain, while the general motion of the atmosphere eastward, in those latitudes, carries this area of greater rarefaction, with its accompanying rain-storm, towards the east, at an average rate of about thirty miles per hour. As the velocity of the general eastward motion of the atmosphere is greater above, the rainy portion of the storm is for the most part on the east side of the line of greatest rarefaction, and as the currents below must be towards this line on both sides, when it passes over any place, the rain generally ceases and the wind changes.

60. When the area of rarefaction is such as to cause the atmosphere to flow in from all sides below towards a centre, and the reverse above, the disturbed portion of atmosphere, if it were not that its motions are resisted by the earth's surface, and the surrounding undisturbed part, would assume the outline and the gyratory motion in the case of no resistance, as represented in Figure 3 and Figure 4. But on account of the resistances, the motions of the atmosphere are very much modified, so that it has only a tendency to assume in some measure those motions, and instead of the atmosphere's receding entirely from the centre on account of the rapidity of the gyrations near the centre as represented in Figure 3, it is only a little depressed in the middle, as represented in Figure 10.

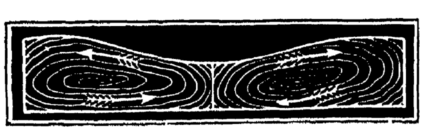

Figure 10.

61. Since the force which produces the gyrations depends upon $\frac{dr}{dt}$, that is, upon the velocity of the flow to and from the centre, it is evident that, at the centre and at the external part of the disturbed portion of atmosphere, where $\frac{dr}{dt}$ must vanish, the resistances destroy all gyratory motion. Hence, instead of very rapid gyrations near the centre, as in the case of no resistances, there must be a calm there, and the most rapid gyrations be at some distance from the centre, in accordance with observation. The diameter of the comparatively calm portion, in the centre of the large cyclones, is sometimes about thirty miles. The velocity of gyration of the external part, which in the case of no resistances, is small, is in a great measure destroyed by the resistances of the surrounding atmosphere, so that it is, for the most part, insensible to observation, and only the more rapid gyrations of the internal part are observed. The motion of gyration combined with the motion at the earth's surface towards the centre, gives rise to a spiral motion towards the centre, exactly in accordance with the observed motions of the atmosphere in great storms or hurricanes, as has been shown by Redfield, in a number of papers on the subject, published in the "American Journal of Science."

62. According to section (29) the gyrations of the inner part of a cyclone must be from right to left in the northern hemisphere, and the contrary in the southern, which is the observed law of storms in all parts of the world, as shown by Redfield, and also by Reid, in his "Law of Storms." It is also evident that at the equator, where cos θ vanishes, there cannot be a cyclone, and hence, of all those which Redfield has investigated, and given in his charts of their routes, none have been traced within $10°$ of the equator. The typhoons or cyclones, also, of the China sea, have never been observed within $9°$ of the equator.

63. That the atmosphere must run into a gyration, if it converge towards a centre, it is evident from the principle demonstrated in section (32), by which, in flowing in from all sides toward the centre, the atmosphere must be deflected to the right in the northern hemisphere, and consequently receive a gyratory motion around that centre, from right to left, and the contrary in the southern hemisphere. Near the equator, this deflecting force vanishes, and consequently there are no cyclones there, as has been shown.

64. Since the atmosphere is depressed in the middle of cyclones, they must sensibly affect the barometer; and this is the true cause of all great barometrical oscillations, as was first suggested by Redfield. As the cyclone approaches, there is generally a very slight rise of the barometrical column, which is at its maximum at the greatest accumulation near the external part of the cyclone, after which it is gradually depressed until the middle of the cyclone arrives, where the atmosphere is most depressed, when the barometer is at its minimum, and then it returns in a reverse manner to its former height, when the cyclone has passed. In great storms, the mercury sometimes falls more than two inches. In oblong storms, and all imperfectly developed cyclones, the same phenomena must take place in some measure, as in a complete cyclone. We have reason to conclude, therefore, that nearly all the oscillations of the barometer are caused by a cyclonic motion of the atmosphere, by which it is depressed in the middle of the cyclones. The cyclones may be very irregular and imperfectly developed, and not of sufficient violence to produce a strong wind, and several may frequently interfere with one another, so that the oscillations may frequently be very slight ones only and very irregular.

Since the gyratory motion of a cyclone, and the consequent depression at the centre, depend upon a term containing as a factor, cos θ, (section 29), which is the sine of the latitude, according to the preceding theory of barometrical oscillation, the oscillations should be small near the equator, and increase towards the poles, somewhat as the sine of the latitude. Accordingly, at the equator, the mean monthly range of oscillation is only two millimeters, or less than $\frac{1}{10}$ of an inch, while there is a gradual increase with the latitude; so that at Paris it is 23.66^{mm}, and at Iceland, 35.91^{mm}. (Kaemtz's Meteorology, by C. Walker, page 297.)

65. The greater rarefaction of the atmosphere at some times than at others, without doubt, has considerable effect upon the barometer; but the theory which attributes the whole of the barometrical oscillations to the rarefaction of the atmosphere produced by the condensation of vapor in the formation of clouds and rain, cannot be maintained; for according to that theory, in the rainy belt near the equator, where there are always copious rains during the day, which are succeeded by clear atmosphere during the night, the oscillations of the barometer should be greatest, and towards the poles, where there is little condensation of vapor into rain, they should be the least, but we have seen that just the reverse of this is true.

66. When the disturbance of equilibrium is great, but extends over a small area only, the centripetal force is much greater than in the case of large cyclones, and the gyrations are then very rapid and very near the centre, as in the case of tornadoes. Tornadoes generally occur when the surface of the earth is very warm and the atmosphere calm. For then the strata near the surface becomes very much rarefied, and are consequently in a kind of unstable equilibrium for a while, when from some slight cause, the rarefied atmosphere rushes up at some point through the strata above, and consequently flows in rapidly from all sides below, and then, unless the sum of all the initial moments of gyration around the centre is exactly equal 0, which can rarely ever be the case, it must run into rapid gyrations near the centre, and a tornado is the consequence. This may be exemplified by the flowing of water through a hole in the bottom of a vessel. If the fluid at the beginning is entirely at rest it runs out without any gyrations; but if there is the least perceptible initial gyratory motion, it runs into very rapid gyrations near the centre.

67. In the case of tornadoes, which are always of small extent, the influence of the earth's rotation in producing gyratory motions is generally very small in comparison with that of the initial

state of the atmosphere, as may be seen by examining equation (42). For if the atmosphere have a very small initial gyratory motion, the term u' depending upon the initial state, will be large in comparison with $n \cos \varphi$ depending upon the earth's rotation, and hence the value of $\frac{d u}{d t}$, the angular velocity of gyration, depends principally upon the initial gyratory state of the atmosphere with regard to the centre of the tornado, and may be either from right to left or the contrary. Hence there may be tornadoes at the equator, although there cannot be large cyclones. In large cyclones the effect of the initial state, except at the equator, is insignificant in comparison with the influence of the earth's rotation; and the latter, moreover, is a constant influence, while the former is soon destroyed by resistances. Hence large cyclones are of long duration, while small tornadoes, depending principally upon the initial gyratory state for their violence, are soon overcome by the resistances.

68. On account of the centrifugal force arising from the rapid gyrations near the centre of a tornado it must frequently be nearly a vacuum. Hence, when a tornado passes over a building, the external pressure, is in a great measure, suddenly removed, when the atmosphere within, not being able to escape at once, exerts a pressure upon the interior of perhaps nearly fifteen pounds to the square inch, which causes the parts to be thrown in every direction to a great distance. For the same reason, also, the corks fly from empty bottles, and everything with air confined within explodes.

69. When a tornado happens at sea, it generally produces a water-spout. This is generally first formed above, in the form of a cloud, shaped like a funnel or inverted cone. As there is less resistance to the motions in the upper strata than near the earth's surface, the rapid gyratory motion commences there first, when the upper strata of the agitated portion of atmosphere have a tendency to assume somewhat the form of the strata in the case of no resistance, as represented in figure 3. This draws down the strata of cold air above, which, coming in contact with the warm and moist atmosphere ascending in the middle of the tornado, condenses the vapor and forms the funnel-shaped cloud. As the gyratory motion becomes more violent, it gradually overcomes the resistances nearer the surface of the sea, and the vertex of the funnel-shaped cloud gradually descends lower, and the imperfect vacuum of the centre of the tornado reaches the sea, up which the water has a tendency to ascend to a certain height, and thence the rapidly ascending spiral motion of the atmosphere carries the spray upward, until it joins the cloud above, when the water-spout is complete. The upper part of a water-spout is frequently formed in tornadoes on land.

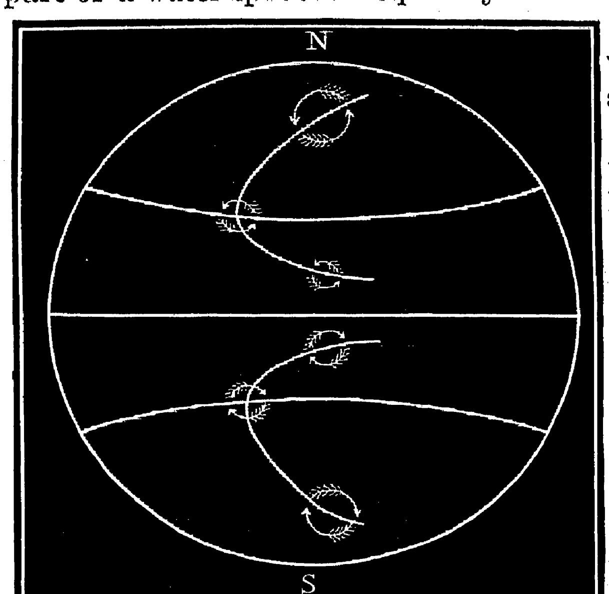

Figure 11.

When tornadoes happen on sandy plains, instead of water-spouts, they produce the moving pillars of sand which are often seen on sandy deserts.

70. The routes of cyclones in all parts of the world, which have been traced throughout their whole extent, have been found to be somewhat of the form of a parabola, as represented in figure 11. Commencing generally near the equator, the cyclone at first moves in a direction only a little north or south of west, according to the hemisphere, when its route is gradually recurvated towards the east, having its vertex in the latitude of the tropical calm-belt, as represented in the figure. This motion of a cyclone may be accounted for by means of what has been demonstrated in (section 31), which is, that if any body, whether fluid or solid, gyrates from right to left, it has a tendency to move toward the north, but if from left to right, towards the south. Hence the interior and most violent portion of a cyclone always gyrating from right to left in the northern hemisphere, and the contrary in the southern, must always gradually move towards the pole of the hemisphere in which it is. While between the equator and the tropical calm-belt, it is carried westward by the general westward motion of the atmosphere there, but after passing the tropical calm-belt, the general motion of the atmosphere carries it eastward, and hence the parabolic form of its route is the resultant of the general motions of the atmosphere, and of its gradual motion toward the pole.

It may be seen from equation (52), that the tendency of a gyrating mass to move towards the

pole is as sin ψ, or the cosine of the latitude, and the square of the diameter of the gyrating mass. Hence, near the equator, where the dimensions of the cyclone are always small, it moves slowly toward the pole, but as it gradually increases its dimensions, after passing its vertex, its motion towards the pole, and also its eastward motion, are both increased, and hence its progressive motion in its route or orbit is then accelerated, in accordance with the observations of Redfield.

71. By comparing equations (27) and (44), it is seen that they are very similar, and consequently the motions which satisfy them must be also similar. Hence the general motions of the atmosphere are similar to those of a cyclone. For the general motions of the atmosphere in each hemisphere, form a grand cyclone having the pole for its centre, and the equatorial calm-belt for its limit. But the denser portion of the atmosphere in this case being in the middle instead of the more rare, instead of ascending it descends at the pole or centre of the cyclone.

The southern cyclone having the more rapid motions on account of the resistances from the earth's surface being less, causes a greater depression of the atmosphere there than in the northern cyclone, and throws the calm-belt a little north of the equator, as has been explained.

The tendency of the smaller local cyclones, as has been seen, is to run into the centres of the grand hemispherical cyclones, and thus to be swallowed up and become a part of them.

SECTION VI.

ON THE MOTIONS OF THE OCEAN.

72. Besides the actions of the sun and moon which give rise to the tides, there are only two causes which can produce any sensible motions on the waters of the ocean. One of these is the action of the atmosphere upon the surface of the ocean, and the other, the difference of density between the water near the equator and that towards the poles, arising from a difference of temperature. The general motions of the atmosphere at the surface of the ocean have a tendency to cause a westward motion of the water in the torrid zone, and an eastward motion in the middle and higher latitudes, and from what we know of the effects of strong winds upon the ocean, we have reason to think that these general motions of the atmosphere are adequate to produce *sensible* motions, since, after the inertia of the water is once overcome, which, however small the force, is only a question of time, the only force necessary is that which is adequate to overcome the resistance of friction, which is very small where the velocity is small. The difference of density between the equator and the poles causes a slight interchanging motion of the water between them, and consequently, where not interrupted by continents, it produces a system of motions in the ocean similar to those of the atmosphere. Hence these two causes of oceanic disturbance, whatever their relative weight, both act in the same directions, and conjointly cause the observed westward motion of the ocean near the equator, and eastward motion toward the poles.

73. The westward motion of the water of the ocean in the torrid zone was first observed by Columbus, and is now well established, and observations also show that there is a motion towards the east in the higher latitudes. A bottle thrown into the ocean near Cape Horn was picked up three and a half years afterward at Port Phillip, Australia, a distance of 9,000 miles, which makes the eastward velocity in that latitude more than seven miles per day. And Sir James Ross, when sailing eastward near Prince Edward's Island, found himself every day from twelve to sixteen miles, by observation, in advance of his reckoning. ("Voyage to the Antarctic Seas," vol. ii, p. 96.) But a westward motion being established in the torrid zone, an eastward motion in the higher latitudes must be admitted; for, as was shown in the case of the atmosphere (section 35), the one cannot exist without the other.

74. It has generally been supposed that the equatorial westward current of the ocean is caused principally by the action of the westward winds there; but Professor Guyot thinks that "it is too deep and rapid to admit of being explained by their action alone," and that "the difference of temperature between the regions near the equator and those near the poles controls all other causes by its power and the constancy of its action." ("Earth and Man," pp., 189, 190.) The torsive or deflecting force which causes the westward motion of the atmosphere and the ocean in the equatorial regions, and the eastward motion in the higher latitudes, has been shown to be as the velocity of the interchanging motion between the equatorial and the polar regions; and hence if this motion in both were similar, the relative amount of this force in each must be as the whole mass multiplied into the velocity of this motion between the equator and the poles. If we suppose the ocean to be three miles in depth, its mass

is about 500 times that of the atmosphere, and hence if the motion between the equator and the poles were only $\frac{1}{500}$ of that of the atmosphere, the part of the force which gives it a westward motion near the equator, and an eastward motion toward the poles, arising from this cause, must be greater than that of the action of the atmosphere upon it, since the whole amount of this force in the atmosphere is not spent upon the ocean, but only that part which overcomes the resistances to its motions. Although the effect of temperature in producing a difference of density, and consequently of disturbing the equilibrium, is very much less in the ocean than in the atmosphere, yet since the amount of motion which a given disturbing force will produce where time is not considered, depends, as has been stated, upon the amount of the resistances, and not upon the amount of inertia to be overcome; and since the resistances diminish as the square of the velocity, a very small amount of disturbing force arising from a difference of density must be adequate to cause an interchanging motion in the ocean between the equatorial and the polar regions equal to $\frac{1}{500}$ of that of the atmosphere; and hence we have reason to think that a greater part of the motions of the ocean is due to this cause than to the action of the atmosphere upon it.

75. The motions of the ocean being similar to those of the atmosphere, they must cause a slight elevation of the surface about the parallels of 30°, and a depression at the equator and the poles, just as in the case of the atmosphere, except that it will be less in the ratio of the relative velocities of the motions of the ocean and of the atmosphere. If we suppose the east and west motions of the ocean to be $\frac{1}{60}$ of those of the atmosphere at the earth's surface, as given in the third column of the computed table (section 48), which would require the maximum eastward velocity in the southern hemisphere to be about ten miles per day, it would cause the surface of the ocean in the southern hemisphere to be about fifteen feet higher at the parallel of 30° than at the pole, and also a little higher than at the equator. Now if the motions which cause this accumulation of water were the same at the bottom of the ocean as at the surface, there would be no tendency of the water to flow out at the bottom from beneath this accumulation; but since the motions there must be much less, it must flow out both toward the equator and the pole, especially toward the latter, as the depression there is much the greater. Since the density of sea-water does not increase below the temperature of 28°, the density of the ocean does not increase beyond a certain latitude, and hence there is no flow of the water at the bottom from the poles toward the equator, arising from the maximum density at the pole, as seems to be the case in a very slight degree in the atmosphere, but the undercurrent at the bottom, arising from the greater pressure about the parallel of 30°, must extend entirely to the poles; so that there must be a slight tendency of the water to rise at the poles, and flow at the surface some distance toward the middle latitudes. As the water toward the bottom of the ocean is always about the same as the mean temperature of the earth, when it first rises to the surface at the pole, it must be much warmer than it is after it has flowed some distance from it, and hence we have reason to think that there may be open polar seas, surrounded by barriers of ice at some distance from the pole, where there is the maximum temperature of the surface water. A surface current from the poles is indicated by the motions of icebergs in both hemispheres from the polar regions towards a lower latitude.

76. Where the east and west motions of the ocean are entirely intercepted by continents, as in the northern hemisphere, the water receives a slight gyratory motion from left to right. The westward motion of the waters of the Atlantic in the torrid zone, impinging against the continent of America, causes the surface of the water of the Caribbean sea and the Gulf of Mexico to be a little above the general level, while the eastward motion of the northern part of the Atlantic causes the surface of the water adjacent to the eastern coast of North America, in that latitude, to be a little lower. Hence, there is a flow of warm water from the Gulf of Mexico along the coast of the United States towards the lower level about Newfoundland, which, on account of the peculiar configuration of the coast about the Gulf of Mexico and the peninsula of Florida, gives rise to the Gulf-stream. The eastward motion, also, of the northern part of the Atlantic causes the surface of the water on the western coast of Europe to be a little *higher* than the general level, while the westward motion in the torrid zone causes it to be depressed on the western coast of Africa a little *below* this level, and hence the water of the eastern side of the Atlantic, flowing from a higher to a lower level, has a motion toward the equator. The whole of the north Atlantic has, therefore, a very slight gyratory motion from left to right, and is supposed to make a complete gyration in about three years.

77. A portion of the equatorial current flowing from the higher level of the Caribbean sea toward

Cape Horn, causes the Brazil current, which is deflected eastward by the general eastward motion of the southern ocean. The east side of the south Atlantic, as well as that of the north Atlantic, seems to have a motion toward the equator. Says Sir James Ross, "There is a current from the Cape of Good Hope along the west coast of Africa sixty miles wide, two hundred fathoms deep, with a velocity of one mile per hour, of the mean temperature of the ocean."—("Voyage to the Southern Seas," vol. ii., p. 35). This cannot be a portion of the Mozambique current from the warm waters of the Indian ocean, passing around the Cape of Good Hope, and giving rise to the equatorial current of the Atlantic, as has been supposed, but must come from the colder waters of the southern ocean. Hence the south Atlantic also has a tendency to assume a gyratory motion, and the equatorial current of the Atlantic is merely the equatorial portion of these two gyrations, with perhaps a small part of the Mozambique current passing around the Cape.

77. The general eastward motion of the water of the northern part of the Atlantic, and the consequent depression of the water next the coast of North America, is the cause of the cold current of water flowing from Baffin's bay and the east coast of Greenland, between the Gulf-stream and the coast of the United States, called the Greenland current. Since the warm water of the Gulf-stream, in flowing northward, is deflected toward the east (section 32), and that of the Greenland current, in flowing south, tends toward the west, there is no intermingling of the waters of the two currents, but they are kept entirely separate as if divided by a wall, as has been established by the Coast Survey.

79. There must be a motion of the water somewhat similar to the Gulf-stream and the Greenland current, wherever the great equatorial current impinges against a continent, and the eastward motion toward the poles is interrupted. Hence, on the eastern coast of South America there is the warm Brazil current which has been mentioned, and on the eastern coast of Asia there is the warm China current, flowing toward the north, similar to the Gulf-stream, and the cold Asiatic current insinuating itself between it and the coast, like the Greenland current. On the east coast of Africa, also, there is the Mozambique current flowing south like the Brazil current, and it is also now well established that, east of the Cape of Good Hope, the general tendency of the water is toward the south. This water must mingle with the general eastward current of the South sea, and hence there is a slight tendency to a gyratory motion in the Indian ocean also.

80. On the western sides of the continents there is a motion somewhat the reverse of this, and instead of a warm current flowing north, there is a cold one flowing toward the equator, as has been shown to be the case in the Atlantic. Hence, on the west coast of North America there is a flow of colder water along the coast from the north, and on the west coast of South America is Humboldt's current, much colder than the rest of the ocean in the same latitude, both tending toward the equator to join the great westward current there across the Pacific, and to fill up, as it were, the vacuum which this current has a tendency to leave about the equator, on the west coast of America.

81. When a portion of fluid on the earth's surface gyrates from left to right, the deflecting force arising from the earth's rotation being in this case toward the interior, the surface assumes a slightly convex form. If, however, the velocity of gyration were equal to twice that of the earth's rotation multiplied by the cosine of the polar distance, the centrifugal force arising from the gyration would be exactly equal to the centripetal force arising from the earth's rotation, and consequently they would neutralize each other, and if the velocity of gyration were still greater, the surface would be convex, as has been shown in section 30. The water of the north Atlantic having a very small gyratory velocity in comparison with that of the earth's rotation, the interior is a little elevated above the general level, and consequently the pressure upon the bottom increased. If we suppose a circular portion of it, 3,000 miles in diameter, with its centre on the parallel of 30°, to perform a gyration from left to right in three years, equation (50) would give an elevation of five feet in the middle above the level of the external part. This equation, however, on account of the term which has been neglected in the analysis (section 25), is not strictly applicable to so large a portion of fluid, but still it gives the order or the effect produced. Now the gyrations which cause this elevation in the middle being principally toward the top, the increased pressure upon the bottom causes the fluid there to flow out on all sides with a very small velocity, towards the circumference, and hence the water at the surface has a slight tendency to flow in from all sides towards the interior to supply its place. This completely accounts for that vast accumulation of drift and sea-weed, covering a large portion of the interior of the north Atlantic, called the Sargasso sea. From what has been stated, the north Pacific must also have a slight gyratory motion from left to right, and hence it likewise has its Sargasso sea.

In section 81, "an elevation of five feet" should be "an elevation of five-tenths of a foot." [W. F.]

SECTION VII.

ON THE MOTIONS OF SOLID BODIES.*

82. It is proposed in this section to treat of the motions of projectiles and rotating bodies relative to the earth's surface, so far principally as they are influenced by the earth's rotation. By putting P and its derivatives in our fundamental equations equal 0, we get the equations of a projectile. Hence we obtain from equations (9), by restoring the value of a in section 2, the following general equations of the motions of a projectile relative to the earth's surface:

$$
(61)\qquad
\begin{aligned}
\frac{d^2 r}{d t^2} &= r\left(\frac{d\theta}{d t}\right)^2 + r \sin^2\theta\left(2 n + \frac{d\varphi}{d t}\right)\frac{d\varphi}{d t} - g,\\
r\frac{d^2\theta}{d t^2} &= -\frac{2\, d r}{d t}\cdot\frac{d\theta}{d t} + r\sin\theta\cos\theta\left(2 n + \frac{d\varphi}{d t}\right)\frac{d\varphi}{d t},\\
r\sin\theta\frac{d^2\varphi}{d t^2} &= -2\sin\theta\left(n + \frac{d\varphi}{d t}\right)\frac{d r}{d t} - 2 r\cos\theta\left(n + \frac{d\varphi}{d t}\right)\frac{d\theta}{d t}.
\end{aligned}
$$

83. Let α, β, and γ be three rectangular co-ordinates of the projectile, of which α is vertical, β directed toward the south, and γ toward the east, and having their origin at the point of projection.

Let u, v, and w be the lineal velocities respectively in the direction of α, β, and γ, and n' the lineal velocity of the rotation of the earth's surface at the latitude of projection.

Also let R, θ', and φ' be the initial values, or the values at the point of projection, of r, θ, and φ, and u', v', and w' the initial values respectively of u, v, and w.

We shall then have, without any sensible error for the limited range of a projectile,

$$
\begin{aligned}
\alpha &= r - R, &\quad \beta &= R\,(\theta - \theta'), &\quad \gamma &= R\sin\theta'\,(\varphi - \varphi'),\\
u &= \frac{d r}{d t}, & v &= R\frac{d\theta}{d t}, & w &= R\sin\theta'\frac{d\varphi}{d t},
\end{aligned}
$$

$$n' = R\, n\sin\theta' = \sin\theta' \times 1523.2 \text{ feet}.$$

By means of these equations, equations (61) are reduced to

$$
(62)\qquad
\begin{aligned}
\frac{d^2\alpha}{d t^2} &= \frac{v^2}{R} + \frac{(2 n' + w)\, w}{R} - g,\\
\frac{d^2\beta}{d t^2} &= -\frac{2 u v}{R} + \frac{(2 n' + w)}{R}\, w\cot\theta',\\
\frac{d^2\gamma}{d t^2} &= -\frac{2\,(n' + w)\, u}{R} - \frac{2\,(n + w)\, v}{R}\cot\theta'.
\end{aligned}
$$

84. In integrating these equations, g, and also v and w, may be regarded as constant, and equal to their initial values.

By a first integration we get, reckoning t from the time of projection,

$$
(63)\qquad
\begin{aligned}
\frac{d\alpha}{d t} &= u = u' - g t + \frac{v^2}{R} t + \frac{(2 n' + w')\, w'}{R} t,\\
\frac{d\beta}{d t} &= v = v' - \frac{2 u' v'}{R} t + \frac{(2 n' + w')\, w'\cot\theta'}{R} t + \frac{2 g v'}{R} t^2,\\
\frac{d\gamma}{d t} &= w = w' - \frac{2\,(n' + w')\, u'}{R} t - \frac{2\,(n' + w')\, v\cot\theta'}{R} t + \frac{2 g\,(n' + w')}{R} t^2.
\end{aligned}
$$

In integrating the last two equations, the value of $u = u' - g t$, neglecting the other two very small terms, was substituted before integration.

By a second integration we get,

$$
(64)\qquad
\begin{aligned}
\alpha &= u' t - \tfrac{1}{2} g t^2 + \frac{v'^2}{2 R} t^2 + \frac{(2 n' + w')\, w'}{2 R} t^2,\\
\beta &= v' t - \frac{u' v'}{R} t^2 + \frac{(2 n' + w')\, w'\cot\theta'}{2 R} t^2 + \frac{2 g v'}{3 R} t^3,\\
\gamma &= w' t - \frac{(n' + w')\, u'}{R} t^2 - \frac{(n' + w')\, v'\cot\theta'}{R} t^2 + \frac{2 g\,(n' + w'}{3 R} t^3.
\end{aligned}
$$

These are the complete equations of a projectile in terms of t and its initial velocity. The terms containing n' arise from the earth's rotation, and those having R in the denominator, from the earth's sphericity.

---

* Many of the results in this section have been given in a special paper on this part of the subject, published in the fifth volume of "Gould's Astronomical Journal." For the sake of completeness, they are here deduced from more general equations and given again.

85. If we neglect the terms depending upon the earth's rotation and sphericity in the preceding equations, they may be reduced to

(65)
$$\alpha = u' t - \tfrac{1}{2} g t^2,$$
$$s = i t,$$

in which s is the horizontal co-ordinate, having for its direction the initial direction of the projection, and i the horizontal velocity of projection. Hence the projectile in this case moves in a vertical plane. These are the equations of a projectile as given in elementary treatises, in which no account is taken of the earth's rotation, and in which the directions of gravity are supposed to be parallel. Eliminating t, we get the equation of a parabola, and hence, in this case, the motion is parabolic.

86. In the more general case, the terms containing n' and R in equations (64) deflect the motion from the parabola, and also from the vertical plane of projection. These terms, however, are very small, and their effects may be regarded merely as small perturbations from parabolic motion.

87. When the initial motion of projection is in a vertical direction, $v' = 0$, and $w' = 0$, and equations (64) become in this case

(66)
$$\alpha = u t - \tfrac{1}{2} g t^2,$$
$$\beta = 0,$$
$$\gamma = -\frac{n' u'}{R} t^2 + \frac{2 g n'}{3 R} t^3.$$

Hence, since $\beta = 0$, an ascending or falling body projected vertically does not deviate to the north or south of a perpendicular; but the value of γ gives the deviation of such a body to the east or west, and depends entirely upon the earth's rotation.

If a body is projected vertically upward, when it arrives at its maximum height, $u = 0$, and the first of equations (63) then gives, neglecting the very small terms, $u' = g t$. Hence the last of the preceding equations gives

(67)
$$\gamma = -\frac{u'^3 n'}{3 R g^2} = -\frac{g n'}{3 R} t^3$$

for the deviation west when the body is at its greatest height.

If a body is let fall from a state of rest relative to the earth's surface, $u' = 0$, and we get in this case

(68)
$$\gamma = \frac{2 g n'}{3 R} t^3$$

for the deviation of a falling body east of the perpendicular.

If a body is projected vertically upward, the last of equations (63) gives, when the body is at its maximum height, putting t' for the time of ascent,

$$w = -\frac{2 n' u'}{R} t' + \frac{2 g n'}{R} t'^2.$$

In applying the equation to the same body in falling, this value of w becomes the initial velocity, and must be substituted for w'. Hence we shall have

$$\frac{d \gamma}{d t} = -\frac{2 n' u'}{R} t' + \frac{2 g n'}{R} t'^2 - \frac{2 n' u'}{R} t + \frac{2 g n'}{R} t^2.$$

Integrating,

$$\gamma = -\frac{2 n' u'}{R} t' t + \frac{2 g n'}{R} t'^2 t - \frac{n' u'}{R} t^2 + \frac{2 g n'}{3 R} t^3.$$

When the body returns to the level from which it is projected, $t = t'$, and $g t = u'$, by which the preceding equation is reduced to

$$\gamma = -\frac{u'^3 n'}{3 R g^2}.$$

It deviates to the west in ascending, we have seen, by the same amount; and hence, if a body is projected vertically upward and returns, it falls to the west of the point of projection a distance equal to twice the preceding value of γ.

All the preceding deviations are small, amounting to only a few inches in a range of several hun-

dred feet. Since the value of n' contains $\sin\theta$ as a factor, they are greatest at the equator and decrease as the sine of the polar distance.

88. When the projectile has a considerable horizontal range, the amount of deflection arising from the earth's rotation is much greater. If the earth had no rotation, the projectile would move in the same vertical plane. The deflection, therefore, from this plane depends upon the terms in equations (64) containing n'. Hence, if $\delta\beta$ and $\delta\gamma$ denote the effect of these terms upon the horizontal co-ordinates, we shall have very nearly, when the projectile does not have a great range in altitude,

$$(69)\qquad \begin{aligned} \delta\beta &= \frac{n'\,w'\cot\theta'}{R}\,t^2 = n\,w'\cos\theta'.\,t^2. \\ \delta\gamma &= -\frac{n'\,v'\cot\theta'}{R}\,t^2 = -\,n\,v'\cos\theta.\,t^2. \end{aligned}$$

The effect of the other terms containing n', we have seen, is very small, unless the range in altitude is great.

Let s be a horizontal co-ordinate in the direction of the horizontal range, and p a co-ordinate at right angles on the right, and i the horizontal velocity of projection. Then, by resolving the preceding co-ordinates and velocities into the directions of s and p, we obtain

$$(70)\qquad p = n\,i\cos\theta'\,t^2.$$

If a cannon-ball were to fly three miles in twelve seconds, the preceding equation, using the value of n in section 18, would give, at the parallel of 45°, about ten feet for the value of p, or the deviation to the right of the initial direction.

89. If a body be supposed to move on the surface of the earth without friction, it would be continually deflected into a curve by the force $\frac{dF}{dp}$ equation (53). If ρ be the radius of curvature, and m the angular velocity about the centre of curvature, the centrifugal force will be $\rho\,m^2$, which must be put equal $\frac{dF}{dp}$. Hence we have from equation (53)

$$\rho\,m^2 = 2\,n\,v\cos\theta.$$

Also, since v is the lineal velocity of the moving body,

$$\rho\,m = v.$$

Hence we have

$$(71)\qquad \begin{aligned} \rho &= \frac{v}{2\,n\cos\theta}, \\ m &= 2\,n\cos\theta. \end{aligned}$$

When the range of motion is so small that $\cos\theta$ may be regarded as constant, ρ and m are constant, and hence the body then moves in the circumference of a circle with a uniform angular velocity. If we put τ for the time of a revolution, we shall have

$$(72)\qquad \tau = \frac{2\,\pi}{m} = \frac{2\,\pi}{n\cos\theta} = \sec\theta \times \tfrac{1}{2}\text{ day}.$$

Hence the time of a revolution is independent of the initial velocity of the body.

90. The gradual gyration of a vibrating pendulum is caused by this same deflecting force, and hence the time of gyration is the same as that of τ in the preceding equation.

91. If a body is forced to move in a straight line, the last form of the value of $\frac{dF}{dp}$, equation (53), serves to compare the lateral pressure of this body with gravity. If we put $v = 60$, which is a velocity of about forty miles per hour, the equation gives $\frac{dF}{dp} = \frac{1}{5188}\,g$, at the parallel of 45°. Hence if a railroad train moves in a straight line forty miles per hour at the parallel of 45°, the lateral pressure on the rails is $\frac{1}{5188}$ of its weight, and this is precisely the same for all directions, and not for the direction of the meridian only, as has been generally supposed.

92. We shall now examine the effect of the earth's rotation upon a rotating body of revolution, so suspended as to be free to turn about any axis of revolution passing through its centre of gravity. The same deflecting forces arising from the earth's rotation which act upon a free body having a

motion relative to the earth's surface, must also act upon the different parts of a rotating body; and unless the sum of the moments of these forces with regard to any axis is equal to 0, it must have a tendency to turn the body around that axis, and consequently change the direction of the axis of rotation, unless it coincides with the axis of the moments.

93. Let A, B, and C be the sums of the moments which tend to turn the rotating body about the axes which are respectively perpendicular to the plane of the meridian, the prime vertical, and the horizon.

Also let r', θ', and φ' be the values of r, θ, and φ belonging to the centre of gravity of the rotating body.

We shall then have

$$\begin{aligned}
A &= \int_m r\,(r - r')\,\frac{d^2\theta}{dt} - \int_m r\,(\theta - \theta')\,\frac{d^2 r}{dt^2}, \\
B &= \int_m r \sin\theta\,(r - r')\,\frac{d^2\varphi}{dt^2} - \int_m r \sin\theta\,(\varphi - \varphi')\,\frac{d^2 r}{dt^2}, \\
C &= \int_m r^2 \sin\theta\,(\varphi - \varphi')\,\frac{d^2\theta}{dt^2} - \int_m r^2 \sin\theta\,(\theta - \theta')\,\frac{d^2\varphi}{dt^2}.
\end{aligned} \tag{73}$$

Let ρ be the distance of any particle of the rotating body from the axis of rotation; α, the distance from the centre of gravity of the plane passing through the particle perpendicular to the axis of rotation; φ, the angle between the plane of the meridian and a vertical plane passing through the axis of the rotating body; ψ, the angle of elevation of the axis above the horizon; i, the angular velocity of rotation; $i\,t + \mu$, the angle between any particle and the vertical plane passing through the axis of rotation. We shall then have

$$\begin{aligned}
r - r' &= -\,\alpha \sin\psi + \rho \cos\psi \cos(i\,t + \mu), \\
r\,(\theta + \theta') &= \alpha \cos\psi \cos\varphi + \rho \sin\varphi \sin(i\,t + \mu) + \rho \sin\psi \cos\varphi \cos(i\,t + \mu), \\
r \sin\theta\,(\varphi - \varphi') &= -\,\alpha \cos\psi \sin\varphi + \rho \cos\varphi \sin(i\,t + \mu) - \rho \sin\psi \sin\varphi \cos(i\,t + \mu).
\end{aligned} \tag{74}$$

In taking the derivatives of r, θ, and φ with regard to t, since φ and ψ change very slowly in comparison with $(i\,t + \mu)$ $\frac{d\varphi}{dt}$ and $\frac{d\psi}{dt}$ may be neglected in comparison with $\frac{d\,(i\,t + \mu)}{dt} = i$. Also r in the last two equations may be regarded as constant. Hence we get

$$\begin{aligned}
\frac{dr}{dt} &= -\,i\,\varphi \cos\psi \sin(i\,t + \mu), \\
r\,\frac{d\theta}{dt} &= i\,\rho \sin\varphi \cos(i\,t + \mu) - i\,\rho \sin\psi \cos\varphi \sin(i\,t + \mu), \\
r \sin\theta\,\frac{d\varphi}{dt} &= i\,\rho \cos\varphi \cos(i\,t + \mu) + i\,\rho \sin\psi \sin\varphi \sin(i\,t + \mu).
\end{aligned}$$

By substituting these values in the right-hand members of equations (61), and then substituting the values of the first members thus obtained, and also the values of the first members of equations (74), in equations (73), and integrating, we get

$$\begin{aligned}
A &= -\,n\,i\,(\sin\theta \sin\psi - \cos\theta \cos\psi) \textstyle\int_m \rho^2, \\
B &= -\,n\,i \cos\theta \cos\psi \sin\varphi \textstyle\int_m \rho^2, \\
C &= -\,n\,i \sin\theta \cos\psi \sin\varphi \textstyle\int_m \rho^2.
\end{aligned} \tag{75}$$

94. If in these equations we put $\varphi = 0$, and $\psi =$ the complement of θ, which is making the axis of the rotating body parallel with the earth's axis, we get $A = 0$, $B = 0$, and $C = 0$; and hence the axis of the rotating body in this position has no tendency to change its direction. In all other positions the small deflecting forces tend to change its direction.

95. If the rotating body is free to turn about any axis, the axis of rotation moves at right angles to the resultant of the forces which tend to change its direction, upon the same principle that the axis of an ordinary gyroscope gyrates horizontally at right angles to the direction of gravity. Hence it must gyrate around the position which it would have, if parallel with the earth's axis. If the body is free to turn about the axis only perpendicular to the meridian, its motion is oscillatory and determined by the first of equations (75). If it is free to turn about the axis only perpendicular to the prime vertical, its motions are determined by the second of equations (74); and if free to turn about

the axis only perpendicular to the horizon, by the last of those equations; in both of which cases the motion is also oscillatory. These forces, however, are so small, even with the most rapid velocity of gyration, and in any experiments with the most delicate apparatus the resistances in comparison are so great, that the preceding motions are not observed, but only a tendency of the axis of the rotating body to assume the position in which all the forces are in equilibrium, which in the general case in which the axis is free to move in any direction, is a position parallel with the axis of the earth.

96. These deductions from theory are in exact accordance with some very delicate experiments made by Foucault with a peculiar form of gyroscope, an account of which is given in the "American Journal of Science and Art," second series, vol. xv, page 263. See also vol. xix, page 141.

SECTION VIII.

CONCLUSION.

97. In the first two sections of the preceding pages we endeavored to determined analytically the motions relative to the earth's surface which would satisfy all the conditions of a fluid surrounding the earth with a rotation on its axis, and also the figure which such a fluid must assume, on the hypothesis that the statical equilibrium of the fluid were very slightly disturbed by a difference of density between the equator and the poles. In the third section we proceeded in a similar manner with regard to a small circular portion of such a fluid, supposing that its statical equilibrium is disturbed by a difference of density between the centre and the external part. All the results obtained are on the hypothesis that the motions of the fluid are not resisted by the earth's surface, or, in the case of a circular portion of the fluid, by the resistance of the surrounding part. In the next three sections the results thus obtained were used to explain the general motions of the atmosphere and the ocean, the variation of barometric pressure at the earth's surface in different latitudes, the motions of cyclones, and the oscillations of the barometer, allowance being made for the modifying effects of friction or resistance from the earth's surface. In the last section the equations for the motions of solids were deduced from the general equations for the motions of fluids, and applied to the determination of the effect of the earth's rotation on the motions of a projectile and of a rotating body at the earth's surface.

98. For the sake of those who are not disposed to enter into a thorough analytical investigation of the subject, we propose to give in conclusion, without regard to quantitative results, a brief explanation of the principal results at which we have arrived analytically, based upon well-known principles, and also to give a few additional items upon several parts of the subject.

99. First, with regard to the general motions of the atmosphere and the ocean, it is well understood that if one part of a fluid has a greater density than another, the difference of pressure causes a current at the bottom from the denser to the rarer part, and a counter current at the top. Hence, if a fluid surrounding the earth is less dense in the equatorial than the polar regions, there must be a current at the earth's surface toward the equator, and a current above toward the poles, unless these motions are modified by other forces. If the earth had no rotation on its axis, this interchanging motion would be always in the direction of the meridians, and consequently there would be no motion of the fluid relative to the earth's surface, east or west. But the earth having a rotation on its axis, by the well-known principle of the preservation of areas, the part of the fluid moving toward the poles, since it becomes nearer the axis of rotation, must acquire a greater angular velocity of rotation than the earth, that is, it must acquire an eastward motion relative to the earth's surface. For the same reason the part moving toward the equator, since it becomes farther from the earth's axis, must acquire a westward motion relative to the earth. Hence it is evident that the atmosphere and the ocean, being less dense at the equator than at the poles, on account of a difference of temperature, must have an eastward motion toward the poles, and a westward motion in the equatorial regions. The velocity of this motion, on the hypothesis that it is not resisted by the earth's surface, and that its initial state was that of rest relative to the earth, is determined by equation (31), and the modifying effect of the resistance of the earth's surface has been treated in the fourth section.

100. If every part of a homogeneous fluid surrounding the earth had the same angular motion of rotation with the earth, it would be in a state of statical equilibrium relative to the earth, having the same elliptical figure or outline, and its pressure upon the earth's surface would be everywhere very nearly the same. The part of the centrifugal force arising from the earth rotation resolved in the

direction of the meridians, which would be necessary to keep it in this state, is evidently $r\,n^2 \sin \theta \cos \theta$. If now any part of the fluid had a greater angular velocity of rotation expressed by $n + \frac{d\,\varphi}{d\,t}$ this force would become $r\left(n + \frac{d\,\varphi}{d\,t}\right)^2 \sin \theta \cos \theta$, and the fluid would press toward the equator with a force represented by the difference of these two expressions, which is $r\left(2\,n + \frac{d\,\varphi}{d\,t}\right) \sin \theta \cos \theta \frac{d\,\varphi}{d\,t}$. If $\frac{d\,\varphi}{d\,t}$ were negative, unless it were greater than $2\,n$, this expression would be negative, and the pressure would be toward the pole. Now since the motion of the atmosphere toward the poles is eastward, and near the equator toward the west, it is evident that this pressure is exerted both from the poles and the equator toward the parallel where the atmosphere has no motion east or west, and consequently must cause an accumulation of the atmosphere there and a greater barometric pressure. The figure which the atmosphere must assume in consequence of these pressures is represented by figure 9.

101. From what precedes, if a body moves on the surface of the earth in the direction of the meridians, it is deflected to the right in the northern hemisphere, and the same is true if it moves in the direction of the parallels of latitude. Now since in any direction in which a body moves its motion must be compounded of these two motions, the resultant of the two deflecting forces belonging to each of these components must cause the body to be deflected to the right, in whatever direction it may move. In the southern hemisphere the deflection must be to the left. This result has been obtained analytically in section 32, and the amount of this deflecting force is given in equation (53).

102. By the preceding principle we may also explain the motion of a cyclone from the equator toward the poles, independently of our analytic results in section 31. The motion of the equatorial side of a cyclone in either hemisphere is always toward the east, and hence the deflecting force causes a pressure toward the equator, but that of the polar side being always toward the west, the deflecting force causes a pressure toward the pole. Now these deflecting forces being as the sine of the latitude, as may be seen from section 100, the pressure on the polar side toward the pole is greater than that on the other side toward the equator, and hence the cyclone moves in the direction of the greatest pressure. It is not to be supposed, however, that there is an actual transfer of all the atmosphere of a cyclone from the equator to the polar regions. For the motions and pressure of the cyclone being greater on the polar side, where the deflecting forces which cause it are greatest, its action upon the atmosphere in advance of it is greater than on the equatorial side, where these forces are much less, and hence new portions of the atmosphere are being continually brought into action on the one side, while the resistance of the earth's surface, and the adjacent portions of atmosphere on the other side, are continually overcoming the comparatively weak forces there, and destroying the gyratory motion of the cyclone; so that the centre of the cyclone is being continually formed in advanced portions of the atmosphere. Since many cyclones are more than one thousand miles in diameter, the difference in the violence of its action on the two sides is very considerable.

103. This same principle may be used to explain some of the motions of a rotating body. Suppose such a body be placed with its axes of rotation in any direction parallel with the horizon. Then the motions of the upper and the lower parts being in contrary directions relative to the earth, and the deflecting force in both being either to the right or the left, according to the hemisphere in which it is, gives the axis of rotation a tendency to assume a perpendicular position. But there are other forces beside these horizontally deflecting forces, so that all the forces which tend to change the position of the axis would not be in equilibrium with the axis in that position. For the equatorial side then would have a motion coinciding in direction with the motion of the earth's rotation, while the other side would have a motion the contrary way, and consequently the centrifugal force arising from the motion of the earth's rotation, combined with that of the rotating body, would be greater on the equatorial than on the polar side, and give the axis of the rotating body a tendency to move in the plane of the meridian. It might be easily shown, when the axis has a position parallel with the axis of the earth, that the forces which tend to change its direction are then in equilibrium, and consequently the axis, if free to turn in any direction, does not change its position.

104. We shall now give a little additional matter upon several points which would more properly have come in the preceding sections. It has been seen (section 100), that, in consequence of the earth's

rotation, the interchanging motion of the atmosphere between the equator and the poles gives rise to a force, represented by $r\left(2n+\frac{d\varphi}{dt}\right)\sin\theta\cos\theta\frac{d\varphi}{dt}$ by which this motion itself is counteracted. For instance, the motion toward the poles in the upper regions causes an eastward motion which gives rise to a force toward the equator, and which, consequently, counteracts the motion toward the poles, and the motion toward the equator produces a westward motion which gives rise to a force acting in the direction of the poles, which counteracts the motion toward the equator. The motion of the atmosphere, therefore, between the equator and the poles, is not produced by the whole force arising from the difference of density between the equator and the poles, but by a small difference only between the two forces. The difference of the preceding forces has been represented by W (section 44), and it was shown in section 49 that it must in general be very small in comparison with those forces themselves. Hence if the earth had no rotatory motion, the force which produces this motion would be very much greater, and there would be a sweeping hurricane from the pole to the equator.

105. The approximate east or west motion of the atmosphere at the earth's surface, and also at the height of three miles, was computed for several parallels of latitude, from equation (60), neglecting the term W, and given in the table in section 48. The computation is based upon small differences of barometric pressure, and also upon the hypothesis that the difference of temperature between the equator and the poles is 60°; and was designed principally to give a general idea of what the motions must be in the upper regions, where the error arising from neglecting W must be the least. The atmosphere is extremely mobile, and consequently is very much disturbed by a great many local and temporary causes, which our investigation does not take into account, so that it is only the average velocity of a very changeable motion which can be compared with our results. There is no determination of such an average for the upper regions of the atmosphere, and even at the earth's surface this average, from the nature of the observations, cannot be accurately determined. It is, however, known from observations of the motions of the clouds, and also of balloons, that the eastward motion is much greater above than at the earth's surface, in accordance with our computed results. In Professor Coffin's "Treatise on the Winds" (Smithsonian Contributions, Vol. VII, p. 184) is a table of resultants determined from observations on the velocity and direction of the wind, from which the average east or west velocity may be determined for comparison with our computed results for the earth's surface. The average of all the observations in England and Scotland gives an eastward velocity of about six miles per hour, and the average of all in the Middle and Western United States gives an eastward velocity only a little more than two miles per hour. The computed velocity, according to our table, for the average latitude of the former observations, is about nine miles per hour, and for that of the latter, about four miles. Hence the computed results are from two to three miles less than those determined by observation, which is evidently the effect of the term W in equation (60), which was neglected. For since the atmosphere has a motion toward the poles in the middle latitudes, W is there positive (section 49), and hence the neglect of it diminishes the results. This difference between the observed and computed results furnishes the value of W for those latitudes at the surface of the earth, and since W must be greatest there, the omission of it cannot cause an error of the same amount in the computed results for the upper regions.

106. With regard to the gyratory motion of the oceans, it may be further added here, that such gyrations are clearly demonstrated by the positions of the isothermal lines, as has been shown by Professor Dana, in a paper read at the twelfth meeting of the American Association for the Advancement of Science (Proceedings, vol. xii., p. 77). According to this paper, the isothermal line of 68° F., in winter, extends, in the North Atlantic, from 56° N., on the American side, to 12° N., on the African, and in the South Atlantic, from latitude 31° S., on the South American coast, to 7° S., on the African side. Similar evidences are given of the gyratory motions, in a less degree, in both the North and South Pacific, and also in the Indian Ocean.

107. The cause of the maximum accumulation of the atmosphere near the parallel of 30°, and the explanation of the southwest winds in the middle latitudes, which are produced by it, were first given in my former essay, published in the year 1856. At the meeting of the British Association for the Advancement of Science, in the year 1857, Mr. Thomson read a short paper in which he explains this accumulation of atmosphere, and the consequent reversion of the lower strata of the atmosphere in the middle latitudes, as arising from the centrifugal force of the eastward motion of the atmosphere, and illustrates the effect of such a force by means of a gyrating vessel of water in which the surface

water recedes from the centre, while at the bottom there is a flowing toward it. In this paper nothing is said of the influence of the earth's rotation, and if he means that the effect is produced simply by the centrifugal force arising from the eastward motion of the atmosphere relative to the earth's surface, independent of the earth's rotation, the force would not be great enough to produce any sensible effect. For by examining the expression of the force which produces this effect, in section 100, it is seen that it depends principally upon the earth's rotation, since $\frac{d\varphi}{dt}$ the angular velocity of the atmosphere relative to the earth, is small in comparison with $2n$, which is double the velocity of the earth's rotation. That the motion of the lower strata only in the middle latitudes is reversed, by this cause, as Mr. Thomson states, and as is represented by figure 9 in this paper, and not the motion of the whole atmosphere in those latitudes, as has been supposed, and as was represented in my first essay, is undoubtedly true; but instead of only a thin stratum next the earth's surface, as Mr. Thomson thinks, observations on the motions of clouds indicate that this stratum extends to a considerable height.

The reader, who has also read my former essay, will notice one or two other slight changes in the results, but all the main points, which were imperfectly set forth in that essay, are fully established by the present more thorough investigation of the subject.

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